
Coxeter Groups and Cellulations

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Abstract

Coxeter groups model symmetries of mathematical objects and provide a connection between reflection groups and tessellations. They intersect the mathematics of geometric group theory, topology, and combinatorics, have inspired the art of M. C. Escher and have found applications in physical and chemical sciences, via simple Lie algebras and crystallography.

This dissertation is an introductory reading for the mathematician interested in studying Coxeter groups. In particular, it explores the geometric implications of Coxeter groups through Coxeter-Dynkin diagrams and Cayley graphs. Finally, it provides a view into Euclidean, spherical and hyperbolic tessellations via Davis complexes Σ and Coxeter cellulations Σ_{cc} .

Declaration

Candidate's Declaration

I, Simone Ivan Conte, do hereby certify that this dissertation, submitted for the degree of MSc, which is approximately 50 pages in length,¹ has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for any degree.

I was admitted as a taught part-time master student at the University of Heriot-Watt in September 2020.

Date

Signature of candidate

¹This includes only the main body of the text. Empty pages are excluded.

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I would like to thank my supervisor Matthew Cordes for his guidance and endless patience throughout the study of this dissertation. I truly enjoyed our conversations as well as rediscovering the beauty of mathematics in Escher's Circle Limit woodcuts.

I am grateful to all the professors I have met throughout my studies at the University of Heriot-Watt for their teachings and feedback, which I will treasure beyond this degree.

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I am forever grateful to my family for their immense and invaluable love and support, which helped me arrive where I am today. I won't stop so far as there is time left.

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Introduction

Repeating patterns have fascinated humans for thousands of years, whether that is a repeating feature in nature, a particular music melody, or just our own reflection on mirrors. Similarly, M. C. Escher's illustrations of impossible geometric figures have mesmerised humans for almost a century. In the 1950s, the Dutch artist wondered whether one could draw an infinite pattern within a finite encircled space.[CEPT86, Cas10] Escher was not a mathematician, at least academically speaking, but he found help in the person of Harold Coxeter and in the mathematics of hyperbolic geometry and Coxeter groups (see Figure 1.1 for some of Escher's work).

While inspired by art, this dissertation is a mathematical introduction to Coxeter groups and cellulations. Our primary goals are: (a) to provide an accessible introduction to group theory, topology, Coxeter groups, and cellulations; (b) to solve exercises from [Sch17] to further our understanding of the subject.

The remaining chapters of this dissertation are:

- Chapter 2 — **Background** — provides a quick introductions to group theory, reflections in geometry, and topology.
- Chapter 3 — **Coxeter Groups** — introduces Coxeter groups.
- Chapter 4 — **Coxeter Cellulations** — explores further Coxeter groups and its cellulations.
- Chapter 5 — **Conclusions** — summarises this dissertation and provides further areas of exploration for the interested reader.



Figure 1.1: M. C. Escher's woodcuts inspired by the work in hyperbolic geometry and Coxeter groups. From left-to-right, top-to-bottom: Circle Limit I (1958), Circle Limit II (1959), Circle Limit III (1959), and Circle Limit IV (1960).

Background

Coxeter groups manifest at the intersection of abstract algebra, geometry, and combinatorics. To appreciate the material presented in the Chapters 3 and 4, we require a basic understanding of these areas of mathematics. We will start with the basic definition of a group and explore the algebraic concepts of generators, actions, and reflections. Then we will quickly look at reflections from a geometric point of view. Finally, we will introduce the concepts of topological spaces and construct CW-complexes and fundamental domains, both relevant objects for the study of Coxeter systems.

The section on group theory is based on the material from [Alu09, DF04, Pin10, SD01]. The section on geometry is based on [Sch17, SD01]. The topology introduction is based on [Hat02, Sch17, SD01, Mor89].

Additional definitions and background material is available in the *Appendix*.

2.1 Group Theory

What do the **integers under addition** and the **set of invertible $n \times n$ matrices under multiplication** have in common? They are both groups!

Integers and matrices look very different to the inexperienced eye. However, they do not to the modern algebraist.² If we ignore the actual values of integers and matrices, and rather focus on how they behave under their named operation, we notice that they both satisfy the same properties of a group.

Definition 2.1.1 (Group). A group is a set G together with a binary operation \circ on G , such that the following axioms are satisfied:

- **(Identity)** $\exists! e \in G$ such that $\forall x \in G$, the following holds: $e \circ x = x \circ e = x$
- **(Associativity)** $(x \circ y) \circ z = x \circ (y \circ z)$
- **(Inverse)** $\forall x \in G$, $\exists! x^{-1} \in G$ such that $x \circ x^{-1} = x^{-1} \circ x = e$
- **(Closure)** The set G is closed under the binary operation \circ , thus if $x, y \in G$, so is $x \circ y \in G$

For simplicity, we may omit the operation symbol \circ going forward. Thus we would write $x \circ y = xy$.³

²This is a bit of a pun from the fact that Abstract Algebra is often called Modern Algebra.

³The operation \circ is sometimes called “the product” or “the addition”, depending on the context. But the reader should always keep in mind that a group can allow any kind of operation as long as the group axioms are satisfied.

Example 2.1. The integers under addition, denoted as $(\mathbb{Z}, +)$, is a group satisfying the four axioms:

- Zero, 0, as the identity e . In fact, any integer added to zero (or vice-versa), is itself.
- Addition is associative as we have learned in our previous years in school.
- The inverse of an integer a is its negative $-a$. Adding an integer to its negative always gives us 0, which is the identity.
- Adding any two integers, always gives us back another integer.

Non Example 2.1. The integers under multiplication, (\mathbb{Z}, \times) , is not a group. This can be proven easily with a counterexample. Take an integer different than zero or one, say 7. Its inverse $\frac{1}{7}$ is not in \mathbb{Z} , thus integers under multiplication do not form a group.

Definition 2.1.2 (Abelian Group). A group is **abelian**⁴ or **commutative** if $x \circ y = y \circ x$ for all x and y in the set G .

Example 2.2. $(\mathbb{Z}, +)$ is abelian since $x + y = y + x$ for any two integers x and y .

Non Example 2.2. $(GL_n(\mathbb{F}), \times)$, the group of $n \times n$ invertible matrices, is non-abelian.⁵ We know from any linear algebra course that for some matrices A and B , the following is true: $AB \neq BA$. We quickly prove this statement with a counterexample:

Proof.

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

□

The identity 0 is clearly “special”, but does the group $(\mathbb{Z}, +)$ have any other elements of interest? Yes, the number 1. If we add it to 0, we get 1. This is trivial, since 0 is the identity. However, we can keep adding it to the result of the previous additions to get all other integers: $2, 3, 4, \dots, n$. Similarly, we can use the *inverse* property of the group to get all negative integers: $-1, -2, -3, \dots, -n$. We call 1 a **generator** of the group $(\mathbb{Z}, +)$ since it allows us to generate any other integer in \mathbb{Z} .

Definition 2.1.3 (Generators). Let S be a set such that $S \subseteq G$, if each element $g \in G$ can be written in terms of $s \in S$ and $s^{-1} \in G$, then we call S the generators of G and we write $G = \langle S \rangle$

Example 2.3. $(\mathbb{Z}, +) = \langle 1 \rangle$

Furthermore, groups may have subgroups. We can think of subgroups as smaller partitions of a group inheriting some of its properties, while still being a group. Visually, one may think of a subgroup as in Figure 2.1.

Definition 2.1.4 (Subgroup). Let H be a set such that $H \subseteq G$. H is a **subgroup** of G if the following conditions hold:

⁴Named after the Norwegian mathematician Niels H. Abel.

⁵ GL stands for General Linear.

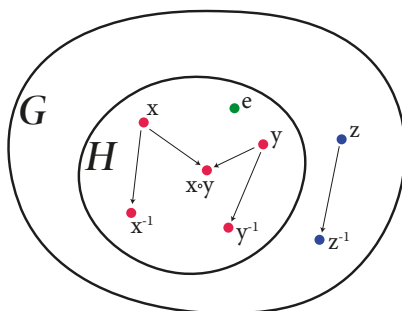


Figure 2.1: Abstract representation of a group and a subgroup.

- $H \neq \emptyset$, since H must have at least one element, the identity e , to be a group.
- H is closed under the binary operation \circ , that is $x \circ y \in H$.
- H is closed under inverses, that is $x \circ y^{-1} \in H$.

We use the notation $H \leq G$ to denote H as a subgroup of G .

Example 2.4. The group of even integers $(2\mathbb{Z}, +)$ is a subgroup of all the integers $(\mathbb{Z}, +)$. Proving this is trivial.

So far we have seen groups with an infinite number of elements, but this does not need to be the case. In fact, when studying Coxeter groups we will mostly be focused on the study of finite groups. That is, groups whose set of elements is finite.

Definition 2.1.5 (Finite Group). A group is **finite** if its set of elements is finite.

Definition 2.1.6 (Order). The **order of a finite group** is the number of its elements. The **order of an element** of a group is the order of the subgroup generated by the element.

Example 2.5. The group $(\mathbb{Z}, +)$ has infinite order. And so does the subgroup of even integers $(2\mathbb{Z}, +)$.

Example 2.6. The group of integers mod 5, denoted $(\mathbb{Z}_5, +)$, has finite order 5.

2.1.1 Morphisms and Relations

When studying mathematical structures we often are interested in understanding their underlying sets and how their elements relate to each other. In particular, we speak of **morphisms**, maps between sets, and **relations**, maps between elements of a set. This section provides a quick overview for some of these concepts. For a better understanding, [Alu09] provides a more formal understanding of these notions from a categorical point of view.

Definition 2.1.7 (Homomorphism). Let (G, \circ) and (H, \star) be groups. Let ϕ be a map $\phi : G \rightarrow H$, such that:

$$\phi(x \circ y) = \phi(x) \star \phi(y)$$

Then ϕ is called a **homomorphism**.

Definition 2.1.8 (Isomorphism). The map ϕ is called an **isomorphism** if it is a bijection.

Definition 2.1.9 (Automorphism). An **automorphism** is an isomorphism from an object to itself.

In group theory, one can think of automorphisms as some kind of “shuffling” of the elements while preserving all the properties of the group. Examples of group automorphisms are given in Section 2.1.2 in the form of permutations.

Elements within a set can have a particular kind of affinity called equivalence relation. We indicate this using the symbol \sim , which should be thought as an abstract placeholder for the actual relation being studied: $<$, $=$, etc.

Definition 2.1.10 (Equivalence Relation). A binary relation \sim on a set X is an **equivalence relation** if and only if all the following axioms are satisfied:

- (**Reflexivity**): $x \sim x$
- (**Symmetry**): $x \sim y \iff y \sim x$
- (**Transitivity**): $x \sim y$ and $y \sim z \implies x \sim z$

Definition 2.1.11 (Equivalence Class). An **equivalence class** for some element a is the class⁶ of all elements $x \in X$ related to a under the equivalence relation \sim :

$$[a] = \{x \in X \mid a \sim x\}$$

It is important to note that if two elements in X are equivalent, $a \sim b$, then each belongs to the equivalence class of the other, $b \in [a]$ and $a \in [b]$. Thus, their equivalence classes are equal: $[a] = [b]$.

Definition 2.1.12 (Quotient with respect to Equivalence). Given a set X , we can “partition” it under some equivalence relation \sim and write:

$$X/\sim$$

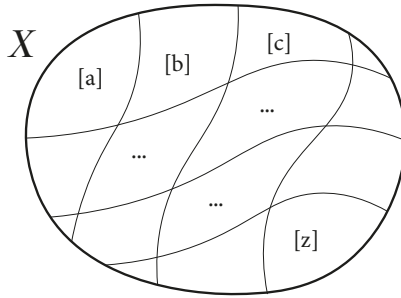
This is the collection of all distinct equivalence classes in X (see Figure 2.2 for an abstract visualisation of the quotient set).

Example 2.7 (The equivalence of even difference). Consider the equivalence relation of integers whose difference is even:

$$x \sim y \iff (x - y) \equiv 0 \pmod{2}$$

There are only two such equivalence classes in \mathbb{Z} : $[0]$ for the even integers (since “even minus even = even”) and $[1]$ for the odd integers (since “odd minus odd = even”). Thus, $\mathbb{Z}/\sim = \{[0], [1]\}$.

⁶For the definition of *class*, see any Set Theory book, such as [Pin14]. However, for the purpose of this text, we can think of a class as a set.

Figure 2.2: Abstract representation of X/\sim .

Definition 2.1.13 (R-Equivalence). Let R be a finite set of words in W_A , the set of all words in the alphabet $A \cup A^{-1}$. Then two words w_1 and w_2 are **R-equivalent** if there exists a finite sequence of reductions or expansions in R leading from w_1 to w_2 . We write $w_1 \sim_R w_2$.

Example 2.8. Let $A = a, b$ and $R = aba^{-1}b^{-1}$. Show that $ab \sim_R ba$.

Proof. Let's start with the word ab and consider the inverse of $aba^{-1}b^{-1}$, which is $bab^{-1}a^{-1}$. Then perform an R-expansion to the left by this inverse on ab , thus having $bab^{-1}a^{-1}ab$. Then reduce $a^{-1}a$, to obtain: $bab^{-1}b$. Reduce again the term $b^{-1}b$ to the identity word, finally obtaining ba . And we are done: $ab \sim_R ba$. \square

2.1.2 Permutations and Symmetric Groups

Let's consider a set of elements S , such that the order of its elements matter. Thus, two sets having the same elements are distinguishable if the order of their elements differ. Provided this, we can define a bijective function from the set to itself:

$$\sigma : S \longrightarrow S'$$

and call it a **permutation**.

The set of all permutations on a set G with the operation \circ being the composition of permutations σ is called the **symmetric group** S_G . Categorically, we would define the symmetric group as the automorphisms on a set G . That would be all the ways we can "shuffle" and "scramble" the set (to itself). However, typically a group theory course would define the symmetric group in the following way.

Definition 2.1.14 (Symmetric Group). A **symmetric group** on a set G is one such that:

- Each element is bijective function from G to G .
- The binary operation \circ is the function composition.

When studying symmetric groups, we do not always care about the nature of its elements. Rather, we are more interested in studying how elements permute under σ . We use the notation S_n to represent any symmetric group of n -elements, where n is a positive integer.

Example 2.9. Let's consider the symmetric group S_4 for the set of four elements: 1, 2, 3, 4.⁷

The identity permutation is defined by:⁸

$$id = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

While the permutation sending each element to the next is define by:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Similarly, we can define a permutation swapping elements in pairs as such:

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Being S_4 a group all properties defined in 2.1.1 apply. For example, $\alpha \circ \beta$ permutes S in the following manner:⁹

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Note that if we allow all permutations to exist, S_n has $n!$ such permutations. Symmetric groups can represent symmetries of geometric shapes too. This will become more relevant in the next chapter, as we study Coxeter groups. In the meantime, we shall look at an example.

Example 2.10. Let's consider the symmetric group S_4 , but this time let's assume its elements are the vertices of a square a, b, c, d . A valid permutation on the square is one that

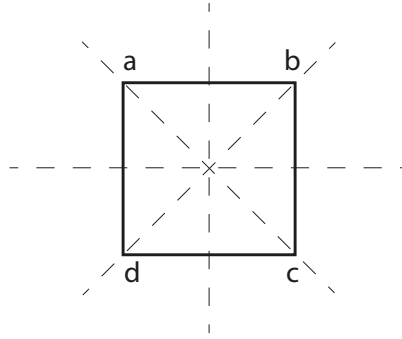


Figure 2.3: Square figure. This can be reflected along the dashed lines or rotated by 90 degrees at a time.

“matches” its previous configuration if we were to unlabel it. Furthermore, the adjacency of the labelled vertices must be preserved.¹⁰

⁷This example works for any set of four elements, such as green, blue, red, yellow.

⁸For simplicity we will represent permutations using Cauchy's two-line notation even though the one line cycle notation is shorter and as common.

⁹Permutations are applied right-to-left. So we apply β first and then α .

¹⁰For example, the vertex **c** must always be adjacent to vertices **b** and **d**. A permutation sending **c** adjacent to **a** and **b** is invalid.

In this particular example, the square can be rotated by 0-, 90-, 180-, and 270- degrees (**rotational symmetries**) and reflected along four axes (**reflection symmetries**), as shown in Figure 2.3. Examples of some permutations are shown in Figure 2.4.

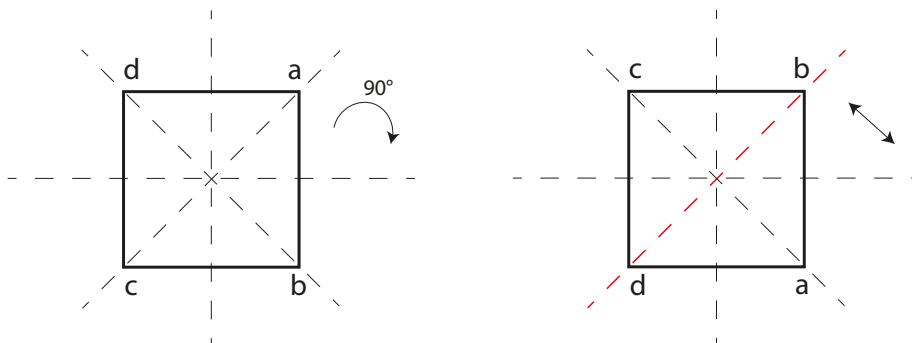


Figure 2.4: Examples of a 90-degree rotation and a reflection along the b-d axis.

Groups representing symmetries on regular polygons are called **dihedral groups**. The group for a regular polygon with n sides is denoted D_n and has $2n$ symmetries: n rotations and n reflections. Thus, the group representing the valid permutations on the square is D_4 and has 8 permutations: four rotations and four reflections. Dihedral groups are Coxeter groups, which we shall study in more details in the next chapter.

2.1.3 Actions

In this section we will study group actions. These are of particular importance to better understand symmetries and transformations. For example, when considering $O(3)$, the group of rotations in the three-dimensional Euclidean space, its actions on regular polygon would describe the rotational symmetries of these geometric objects.

Definition 2.1.15 (Action). Let G be a group and X be a set. Then a group action ϕ is defined as the function:

$$\phi : G \times X \longrightarrow X$$

such that the following two conditions are satisfied:

- **(Identity)**: $\phi(e, x) = x$
- **(Compatibility)**: $\phi(g, \phi(h, x)) = \phi(g \circ h, x)$

where $g, h \in G$ and $x \in X$.

For brevity, we omit the ϕ symbol and simply write $g \cdot x$ for an element g action on x .

Example 2.11. Dihedral groups act on regular polytopes as shown in Example 2.10, via rotations and reflections.

Proposition 2.1.1. A group action $\phi : G \times X \longrightarrow X$ is equivalent to the homomorphism $\psi : G \longrightarrow S_X$, where S_X is the symmetry group on the set X .

Proof. We break the proof in two parts: \implies and \impliedby .

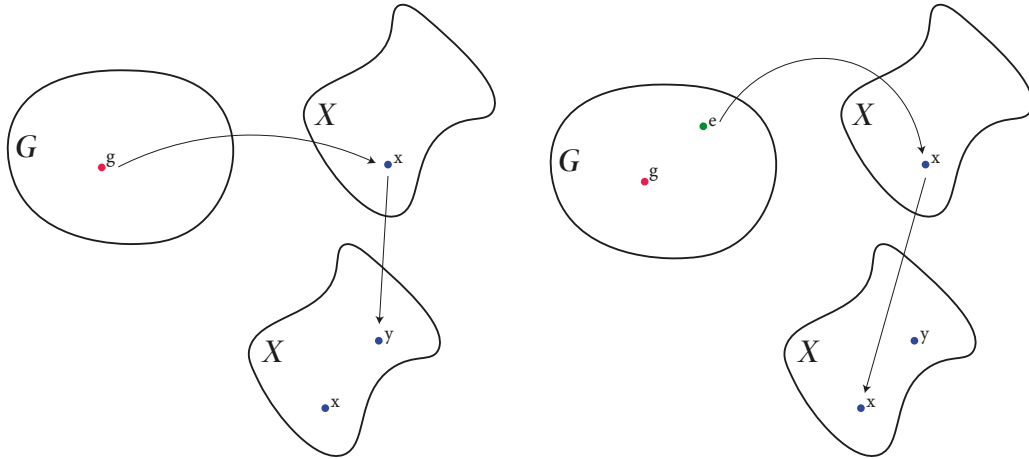


Figure 2.5: Abstract representation of a group action (left) and the identity action on some element $x \in X$ (right).

\implies .

(Identity): Given that ϕ is an action, we have that $\phi(e, x) = x$, but x can be viewed the same as applying the identity permutation id on x , that is $x = id(x) = \psi(e)(x)$. Therefore $\phi(e, x) = \psi(e)(x)$.

(Compatibility): On the other hand, when applying a group action on combined elements from G , we have that $\phi(g, \phi(h, x)) = \phi(g \circ h, x) = \psi(g \circ h)(x)$ and by homomorphism on ψ , $\psi(g \circ h)(x) = \psi(g) \circ \psi(h)(x)$. Thus $\phi(g, \phi(h, x)) = \psi(g) \circ \psi(h)(x)$. \square

\impliedby .

(Identity): First of all, we look at the behaviour of ψ on the identity: $\psi(e)(x) = id(x) = x = \phi(e, x)$, where id is the identity permutation. Therefore $\psi(e)(x) = \phi(e, x)$.

(Compatibility): Then we look at ψ on combinations of elements: $\psi(g) \circ \psi(h)(x) = \psi(g \circ h)(x)$ by the homomorphism property of ψ . Then $\psi(g \circ h)(x) = \phi(g \circ h, x) = \phi(g, \phi(h, x))$. Thus $\psi(g) \circ \psi(h)(x) = \phi(g, \phi(h, x))$. \square

This concludes the proof for the proposition. \square

2.1.3.1 Orbits and Stabilizers

Previously, we have seen how the quotient on an equivalence relation is a mechanism for partitioning a set. Similarly, we can study actions and their orbits to partition sets into equivalence classes.

Definition 2.1.16 (Orbit). Let G be a group acting on a set X . The **orbit** of an element $x \in X$ is:

$$\text{Orb}(x) = \{y \in X : y = g \cdot x\}$$

for some $g \in G$.

On the other hand, we also wish to understand the symmetries that fix an element in the set. These actions are stabilizers and are associated to the definition below.

Definition 2.1.17 (Stabilizer). Let G be a group acting on a set X . The **stabilizer** of an element $x \in X$ is:

$$\text{Stab}(x) = \{g \in G : x = g \cdot x\}$$

Example 2.12. Consider the symmetries of the group S_4 and the set $X = \{1, 2, 3, 4\}$. Then $\text{Stab}(1)$ is the set of all permutations that keep the element 1 in position. The *id* permutation is trivially in $\text{Stab}(1)$. The following is another permutation in $\text{Stab}(1)$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

Definition 2.1.18 (Proper Action). An action is **proper** if $|\text{Stab}(x)| < \infty; \forall x \in X$.

2.1.4 Group Presentations and Cayley Graphs

A group can be described in terms of its generators and relations using a **group presentation**:

$$\langle S \mid R \rangle$$

We recall the definition of generators from Definition 2.1.3. A **relation** is an equality that the generators must satisfy.

Example 2.13. The dihedral group D_3 , the group of symmetries of an equilateral triangle, has the group presentation:

$$\langle r, s \mid r^3 = e, s^2 = e, sr sr = (sr)^2 = e \rangle$$

We may omit the equalities ($= e$) in R and write more concisely:

$$\langle r, s \mid r^3, s^2, (sr)^2 \rangle$$

Finite groups can also be visualised using **Cayley graphs** or **diagrams**. We will use the construction rules from [Pin10]:

- There is one vertex for every element of the group.
- Vertices are joined by arrows.
- Arrows represent the result of multiplying by a generator.
- Arrows are coloured differently for each generator (*e.g.*, red for s , black for r , ...).

Example 2.14. The group D_3 has two generators, let's call them r and s as in the presentation above. Thus its Cayley diagram requires one arrow for each generator. We begin its construction from the identity element e composed with each of the generators (see Figure 2.6a). If we keep applying the generators, while keeping in mind the identities from our relators in R , then we obtain the Cayley graph from Figure 2.6b. Note that double ended arrows represent reflections of order 2: $s^2 = e$. Furthermore, traversing an edge in the opposite direction of its arrow, is the same as applying the inverse of the generator: s^{-1} or r^{-1} .

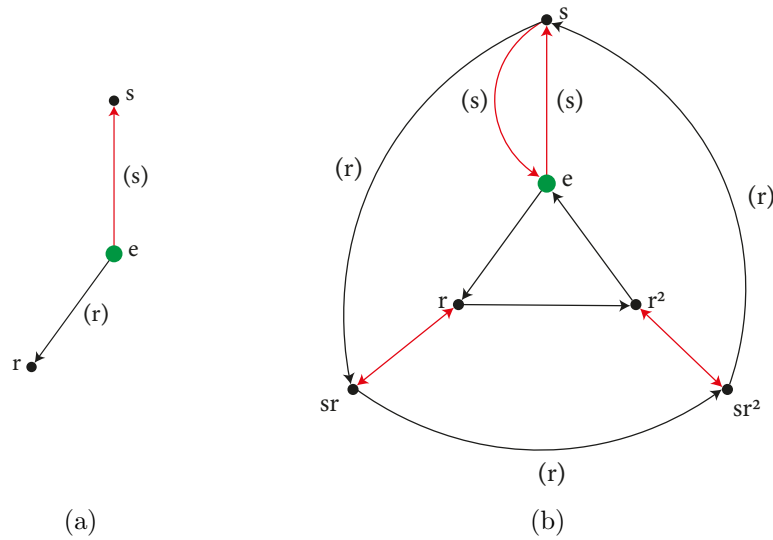


Figure 2.6: (a) Beginning of drawing the Cayley graph of D_3 . Letters in parenthesis are aids to show the generator as part of the applied operation on the starting node. (b) Cayley graph of D_3 .

Example 2.15. For the group D_4 , we recall the Example 2.10 and again note that dihedral groups have two generators r and s . Its presentation then is:

$$\langle r, s \mid r^4, s^2, rsrs \rangle$$

Having the presentation of the group, we can draw the Cayley diagram as we have done in the previous example (see Figure 2.7).

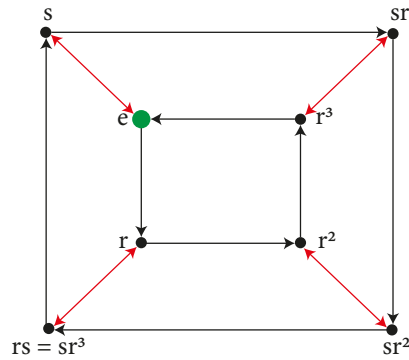


Figure 2.7: Cayley graph for D_4 .

However, not all groups require as many relations. Thus one may wonder: what is the “freest” group of them all?

Definition 2.1.19 (Free Group). A group F with generators S is a **free group** if the only relations are the ones required by the definition of a group itself.

$$\langle S \mid \emptyset \rangle$$

or more concisely:

$$\langle S \rangle$$

We use the notation F_n to indicate the free group with n generators. The free group is the one allowing the generation of words of any length: $aababbbbaaabb\dots$

Example 2.16. The group $(\mathbb{Z} * \mathbb{Z}, +)$ has two generators $a, b \in \mathbb{Z}$ and it is a free group: F_2 . If we think of the addition in terms of concatenation, then operations on a and b form words. An example of a word would be $abaab = aba^2b$. Generators have inverses too: a^{-1} and b^{-1} , so that pairs aa^{-1} and $a^{-1}a$ are reduced to the empty word (and similarly for b). An example of reduction is: $abb^{-1}a^2 = aba^2$.

The Cayley graph of F_2 is:

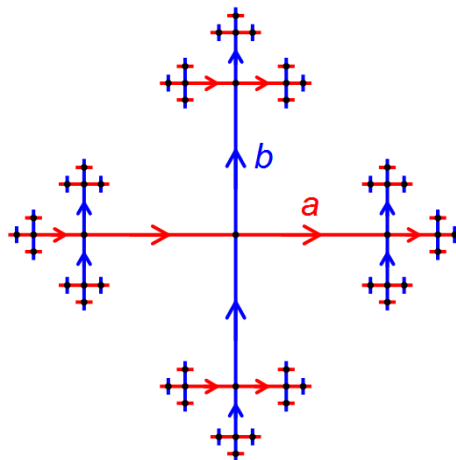


Figure 2.8: Cayley graph of F_2 .

Non Example 2.3. The dihedral group D_n is not free, since its relations, **rotations** ($r^n = e$) and **reflections** ($s^2 = e$), produce a finite number of elements in the group and thus the group is finite.

2.2 Reflections in Geometry

Reflections are a fascinating phenomenon in the physical world. Who hasn't played with one or more mirrors in their childhood to see how object would reflect and distort. From a mathematical perspective, it gets even more interesting as we can experience reflections in low or very high-dimensional Euclidean spaces. But we could also have reflections in spaces with negative curvatures, hyperbolic spaces, or with positive curvatures, spherical spaces. For the purpose of this chapter and the ones that follow we will focus mainly on Euclidean reflections and only briefly look at hyperbolic ones without going too much into the details of its geometry.

Our main interest will be on isometries, distance preserving maps, but first we shall define formally metric and metric spaces.^{11,12}

¹¹These are usually defined in a functional analysis or geometry course, so for brevity reasons examples about different metrics and metric spaces will not be provided.

¹²The study of groups via geometric and topological objects, such is the study of Coxeter groups, is

Definition 2.2.1 (Metric). A **metric** is a function $d : X \times X \rightarrow \mathbb{R}$ which measures the distance between two points $x, y \in X$, such that the following axioms are satisfied:

- **(Point equality)**: $d(x, x) = 0$
- **(Positivity)**: $x \neq y \implies d(x, y) > 0$
- **(Symmetry)**: $d(x, y) = d(y, x)$
- **(Triangle inequality)**: $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.2.2 (Metric Space). A set X endowed with a **metric** is a **metric space**.

Definition 2.2.3 (Isometry). A metric preserving function is an **isometry**.

Intuitively, we can think of an isometry as a transformation (*e.g.*, rotation, translation, reflection) on an object that keeps it intact. In particular, isometries by reflections are of interest to us. These can be thought as mirrors that do not distort objects when we look at their reflections. This property holds on the metric space we are working on, whether it is a 1-dimensional Euclidean line or a 2-dimensional hyperbolic plane.

2.2.1 Reflections in Euclidean space: \mathbb{E}^n

In the 1-dimensional line, we can have reflections across a given point. For example, if we assign to the point the value zero, we can think reflections in an algebraic manner and have them act as inverses. Similarly, in \mathbb{E}^2 , we could have reflections across the y -axis defined by the map: $(x, y) \rightarrow (-x, y)$.

More generally, reflections across any line (in vector notation) are given by (see Figure 2.9):

$$r(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{u}, \mathbf{x} - \mathbf{x}_0 \rangle \mathbf{u}$$

Where:

- \mathbf{x} is the point being reflected.
- x_0 is the point on the line of reflection.
- \mathbf{u} is the unit vector in the direction of the line.
- \langle, \rangle is the dot product of two vectors.

We also note that reflections exhibit symmetry and thus are of order two, meaning that applying a reflection twice give us back the original geometric object in space. In the next chapter, when looking at Coxeter groups, we will study how reflections can be combined so that they have order greater than two.

2.2.2 Reflections in Spherical space: \mathbb{S}^n

A spherical space \mathbb{S}^n is one with a positive curvature and such that the sum of angles within a triangle is greater than π . Reflections work similarly to the Euclidean space:

$$r(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{u}, \mathbf{x} \rangle \mathbf{u}$$

such that $\langle x, x \rangle = 1$, as all points in the space live on the sphere.

known as **geometric group theory**.

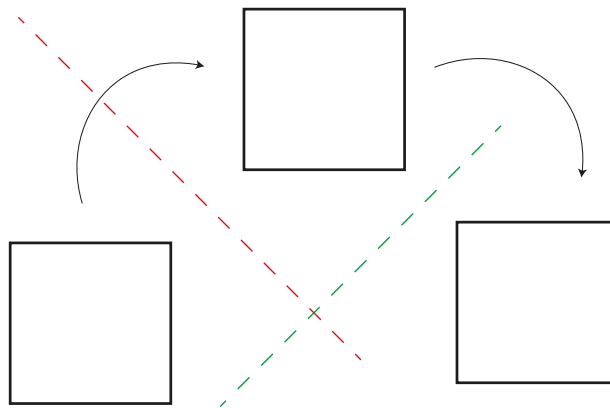


Figure 2.9: Examples of reflections in \mathbb{E}^2 along two arbitrary lines (the red dashed line and the green dashed line).

2.2.3 Reflections in Hyperbolic space: \mathbb{H}^n

Hyperbolic geometry is a fascinating and extensive area of mathematics, of which we are forced to provide a short and incomplete summary due to space constraints. We recall from our geometry studies, that the Euclidean space has a zero curvature, meaning that the plane is flat (and similarly in n -dimensional spaces). In particular, we will reference two important facts from [CFK⁺97]:

Fact 2.1. Given a line and a point not on it, there is more than one line going through the given point that is parallel to the given line.

Fact 2.2. Triangles in hyperbolic space have an angle sum less than π .

As a consequence of this hyperbolic spaces are “very spacious”. For example, it allows us to pack more than four right-angles on a plane. Due to this nature of the hyperbolic space, it is not possible to represent it in an Euclidean plane without allowing for some deformations.¹³ Thus we are forced to use an adapted definition of the inner product:

Definition 2.2.4 (Minkowski’s inner product in \mathbb{R}^3).

$$\langle \mathbf{x}, \mathbf{y} \rangle_M = x_1y_1 + x_2y_2 - x_3y_3$$

Considering the hyperbolic space \mathbb{H}^2 , such that $\langle \mathbf{x}, \mathbf{x} \rangle_M = -1$ and $x_3 > 0$, we can determine the reflections on lines in the space as:

$$r(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{u}, \mathbf{x} - \mathbf{x}_0 \rangle_M \mathbf{u}$$

Similar to how we require some kind of deformation when visualising the world map on a flat plane, so we do for hyperbolic spaces. A mechanism to visualise \mathbb{H}^2 is via the **Poincaré disc**, where we are able to keep angles faithful but must distort distances between points. The Poincaré disc enables us represent the space as a unit disc such that:

¹³Over the last 30 years, the mathematician Daina Taimina has done groundbreaking work on improving our ability to visually understand hyperbolic spaces via crocheting. The interested read should explore some of her work and talks available on the internet.

- Every point in the hyperbolic space \mathbb{H}^2 is contained in the disc.
- Straight lines in the hyperbolic space are drawn as curves that touch the boundary of the disc orthogonally (at a right angle).

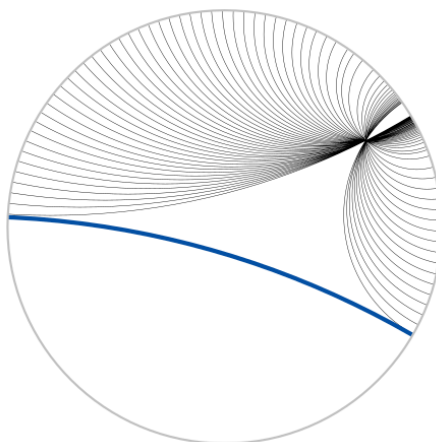


Figure 2.10: Hyperbolic parallel lines in a Poincaré disc.

2.3 Topology

We conclude this chapter with an overview of fundamental concepts in topology that will be of relevance in the chapters following and in particular when looking at cellulations in Chapter 4.

We start by defining the topology on a set as following:

Definition 2.3.1 (Topology). Let X be a set and τ be a collection of open subsets of X . Then τ is the **topology** on X if the following axioms are satisfied:

- X and \emptyset are both elements of τ .
- The finitely or infinitely union of any elements in τ is an element in τ .
- The finitely intersection of many elements in τ is an element in τ .

Definition 2.3.2 (Topological Space). The pair (X, τ) is called a **topological space**. For simplicity, we often omit τ and just say that X is a topological space.

Famously, topology is explained to newcomers as the subject unable to differentiate a coffee mug and doughnut. However, we wish to be mathematically rigorous and in order to establish that two shapes or topological spaces are “similar”, we require the definition of homeomorphism.

Definition 2.3.3 (Homeomorphism). A function $f : X \rightarrow Y$ between two topological spaces X and Y is an **homeomorphism**¹⁴ if the following conditions hold:

¹⁴Note that while the words homomorphism and homeomorphism are quasi-homophones, their definitions differ.

- f is a continuous function.
- f is bijective.
- f^{-1} is also a continuous function.

In other words, homeomorphism allows us to describe continuous deformations, via stretching and bending but not cutting, that allow us to preserve geometric structures. So while a coffee mug is a doughnut, a doughnut is not a mug plate.

Definition 2.3.4 (Actions on Topological Spaces). A group **action** on a topological space X is defined by the homomorphic map:

$$\phi : G \longrightarrow \text{Homeo}(X)$$

where G is the group and $\text{Homeo}(X)$ is the set of homeomorphisms on the topological space X .

Definition 2.3.5 (Quotient Topology). Consider the quotient set X/\sim , as defined in 2.1.12. The **quotient topology** is a topology on X/\sim for which the map $q : X \longrightarrow X/\sim$ is continuous.

Examples about quotient topologies can be found in [Mor89].

2.3.1 Topological Complexes

In topology, complexes are topological spaces composed of simpler structures, known as simplices, that are “glued” together via some triangulation process. There are different gluing mechanisms that can generate such complexes, however for the interest of this work we will look at two such cases: simplicial complexes and CW-complexes.

Definition 2.3.6 (Simplex or Skeleta). A **simplex** is the generalised geometric object for the triangle in n -dimensions. The k -simplex is an object with $(k + 1)$ vertices such that:

- $k = 0$, the simplex is a point.
- $k = 1$, the simplex is a segment.
- $k = 2$, the simplex is a triangle.
- $k = 3$, the simplex is a tetrahedron.
- And so on in higher dimensions.

A **face** of a simplex is a simplex f generated from a subset of its vertices. For instance, a triangle, has three 1-faces (*i.e.*, three segments) and three 0-faces (*i.e.*, three points).

Definition 2.3.7 (Simplicial Complex). A **simplicial complex** is a collection of simplices such that:

- If the complex contains a simplex k , then each of its faces is also contained in the complex.
- The intersection of two simplices k_1 and k_2 is a face of both k_1 and k_2 .

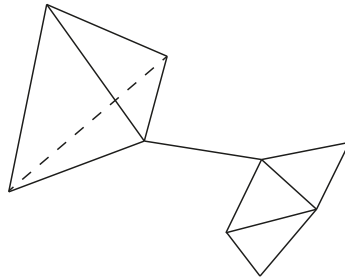


Figure 2.11: Example of a simplicial complex created from one 1-skeleta, three 2-skeleta, one 3-skeleta.

- Simplices are “glued” together in increasing order of dimension (*i.e.*, points first, then segments, then triangles, etc...).

Example 2.17. Simplicial complex for S^2 requires the following steps:

- Take four 0-simplexes
- Take six 1-simplexes
- Glue each 1-simplex to a pair of 0-simplex to form the 1-skeleton of a tetrahedron
- Take four 2-simplexes and glue them to the 1-skeleton to form an empty tetrahedron with filled surfaces

This process is shown in Figure 2.12 and the resulting space is homeomorphic to S^2 .

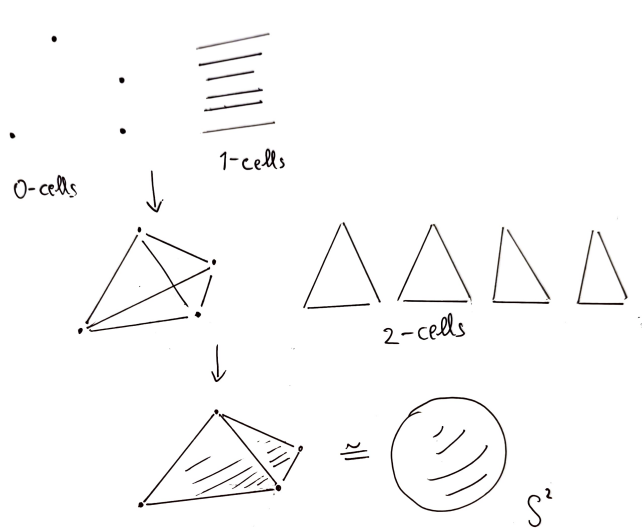


Figure 2.12: Simplicial cellulation of S^2 .

In order to build a simplicial complex we perform a triangulation of the space so that any two triangles do not share more than one edge at once. As a result, the construction of non-trivial spaces using simplicial complexes can be challenging and time consuming.

However, similar results can be achieved by relaxing some of its properties and obtain topological spaces with less structure. In particular, we want to allow “gluing” of multiple faces together and also be able to “glue” any two n -skeleta to the same vertices.

Definition 2.3.8 (CW-complex). A CW-complex is an n -dimensional topological space built from simpler topological spaces of dimensions up to n .¹⁵ These simpler spaces, also called the “building blocks” or **cells** of the complex, are topological n -balls, which are defined as usual:

$$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

The boundary of the n -ball is indicated by ∂D^n and it is the $(n - 1)$ sphere:

$$S^{n-1} = \{x \in \mathbb{R}^{n-1} : \|x\| = 1\}$$

Unlike the simplicial complex, we allow n -balls to be of any geometric figure. So a disc is equivalent to any non-convex 2-dimensional shape, such as a square. Similarly, a ball is equivalent to a cube. And so on at each dimension. The process of gluing cells together is called **cellulation**. In Chapter 4 we will use the terms **mirror-** and **Coxeter-**cellulations for Davis complexes.

Definition 2.3.9 (Finite vs. Infinite). A complex is **finite-dimensional** if there are a cells up to a finite dimension. That is, building the complex requires a finite number of iterations. A complex is finite if it has a finite number of cells, otherwise it is infinite.

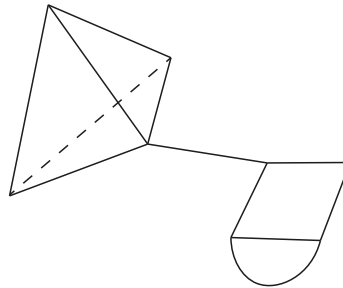


Figure 2.13: Example of a CW-complex created from one 1-skeleta, two 2-skeleta, one 3-skeleta.

Example 2.18 (Cellulation of S^2). One way to create a CW-complex for the 2-sphere, is by starting with a point, the 0-cell, and then “glue” the boundary of a disc, the 2-cell, to it. One can imagine this as wrapping a piece of round cloth in one hand as to form a sack. See Figure 2.14 for a graphical representation of this cellulation.

Example 2.19 (Cellulation of T^2). The construction of a CW-complex for the 1-genus Torus can be done as such:

- Take one 0-cell
- Take two 1-cells and glue all their endpoints to the 0-cell. Identify each 1-cell distinctly.

¹⁵C stands for closure-finite and W for weak topology.

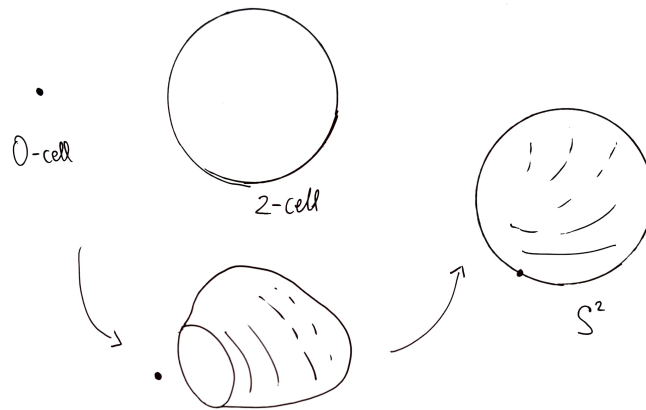


Figure 2.14: CW cellulation of S^2 .

- Take a rectangular shaped 2-cell and identify its sides alternatively. Glue its sides to the 1-cells following the identification prior.

This process is illustrated in Figure 2.15 and the resulting space is homeomorphic to T^2 .

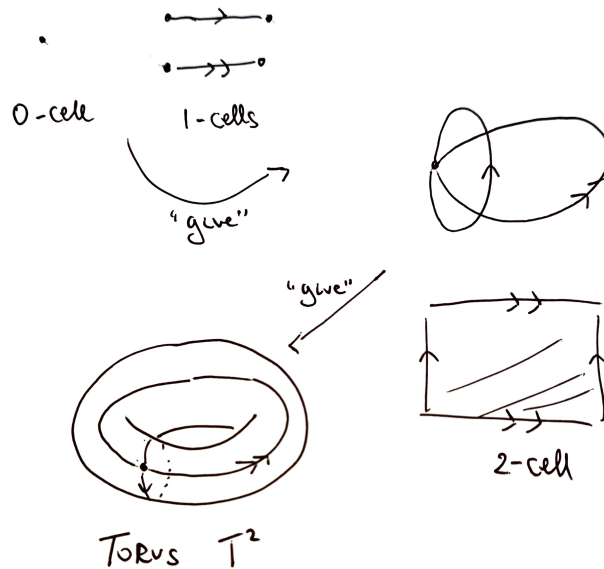


Figure 2.15: CW cellulation of T^2 .

Definition 2.3.10 (Cellular Actions). Let G be a group and X be a CW-complex made of n -dimensional cells σ . A **cellular action** ϕ is defined by:

$$\phi : G \times X \longrightarrow X$$

$$\phi : g \cdot x$$

Thus the cellular action is one that maps m -cells to m -cells for all $0 \leq m \leq n$.

The definitions of stabilizers and orbits are equivalent to the ones from Section 2.1.3.1, but actions act on n -cells of a CW-complex.

We quickly recall the definition of the Euler Characteristic from classical geometry.

Definition 2.3.11 (Euler Characteristic for Polygons). The Euler Characteristic χ is defined in classical geometry as: $\chi = V - E + F$.¹⁶

χ gives us a classification of geometric objects. For example, we know from our earlier studies that the Euler Characteristic for all simply-connected 2-dimensional polygons (*e.g.*, triangle, square, pentagon, ...) is one. Topology teaches us that for any n -dimensional topological space we can extend our definition of the Euler characteristic as such:

Definition 2.3.12 (Euler Characteristic). The Euler Characteristic χ of complexes is defined as $\chi = f_0 - f_1 + f_2 - \dots$, where f_i is the number of cells¹⁷ for dimension i .

Example 2.20. The Euler characteristic for the empty tetrahedron (see Figure 2.16) is:

$$\chi = 4 \text{ (vertices)} - 6 \text{ (edges)} + 4 \text{ (faces)} = 2$$

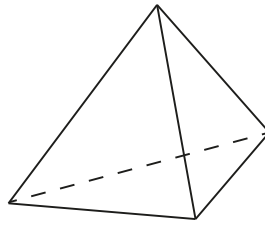


Figure 2.16: Complex for a tetrahedron.

Example 2.21. The Euler characteristic for the empty cube (see Figure 2.17) is:

$$\chi = 8 \text{ (vertices)} - 12 \text{ (edges)} + 6 \text{ (faces)} = 2$$

Example 2.22. The Euler characteristic for the sphere S^2 can be calculated by counting the skeletons in each dimension for its CW-complex (recall Example 2.18):

$$\chi = 1 |0\text{-skeleton}| - 0 |1\text{-skeleton}| + 1 |2\text{-skeleton}| = 2$$

Example 2.23. The Euler characteristic for the torus T^2 can be calculated by counting the skeletons in each dimension for its CW-complex (recall Example 2.19):

$$\chi = 1 |0\text{-skeleton}| - 2 |1\text{-skeleton}| + 1 |2\text{-skeleton}| = 0$$

Definition 2.3.13 (Co-compactness). An action of a group G on a complex X is **co-compact** if the quotient space X/G is a finite complex.[Sch17]

¹⁶Recall from geometry that V is the number of vertices for the geometric figure, E is the number of edges, and F is the number of faces.

¹⁷Cells are also called faces in some textbooks.

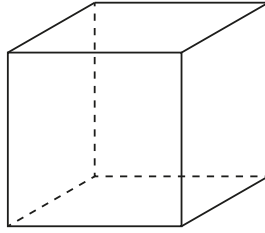


Figure 2.17: Complex for a cube.

2.3.2 Fundamental Domain

In our study of Coxeter groups, we will investigate how cells “reflect” under particular conditions. In order to do so we will use a **strict fundamental domain** as our unit block to do so.

Definition 2.3.14 (Fundamental Domain). Let G be a group acting on a topological space X . Consider $orb(x)$, the orbit for some $x \in X$. The **fundamental domain** is the set $D \subseteq X$ made of the union of all orbits such that X is contained in it.

This definition is not trivial, so let’s unpack it through some examples and visuals.

Example 2.24 ($(\mathbb{Z}, +)$ acting on \mathbb{R}). Let’s consider the real line acted upon by the group of integers under addition.

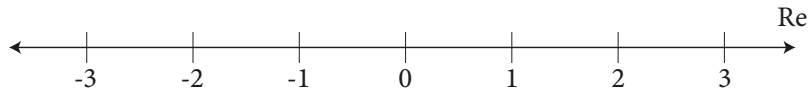


Figure 2.18: The real line.

The actions of \mathbb{Z} on elements of \mathbb{R} are: $x + z$, for $x \in \mathbb{R}$ and $z \in \mathbb{Z}$. For example, for $z = 1$ is equal to any real number plus one. Similarly for $z = -1$, we get any real number minus one. And so on.

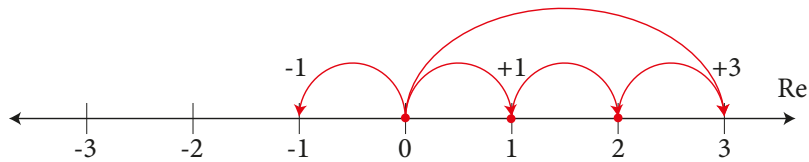


Figure 2.19: Examples of actions of $(\mathbb{Z}, +)$ on the real line.

We can see from Figure 2.19 how the action of each z determines its own orbit on \mathbb{R} . If we consider the interval $[0, 1] \in \mathbb{R}$, we can observe how the orbit of each element in \mathbb{R} is identified with a corresponding element in successive intervals by translation. The interval $[0, 1]$ is our fundamental domain.

Example 2.25 ($(\mathbb{Z} * \mathbb{Z}, +)$ acting on $\mathbb{R} \times \mathbb{R}$). The previous example can be extended for $(\mathbb{Z} * \mathbb{Z}, +)$ acting on the 2-dimensional Cartesian plane \mathbb{R}^2 .

In this case, our group has two distinct generators. Visually, we can think of these as horizontal and vertical translations in the plane. In this case, our fundamental domain

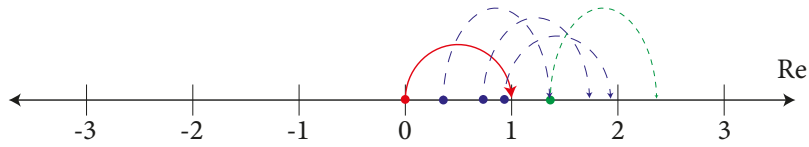


Figure 2.20: Example of how orbits determine the fundamental domain.

$[0, 1] \times [0, 1]$ is a unit-square that can be translated around to form a grid of unit squares, as shown in Figure 2.21.

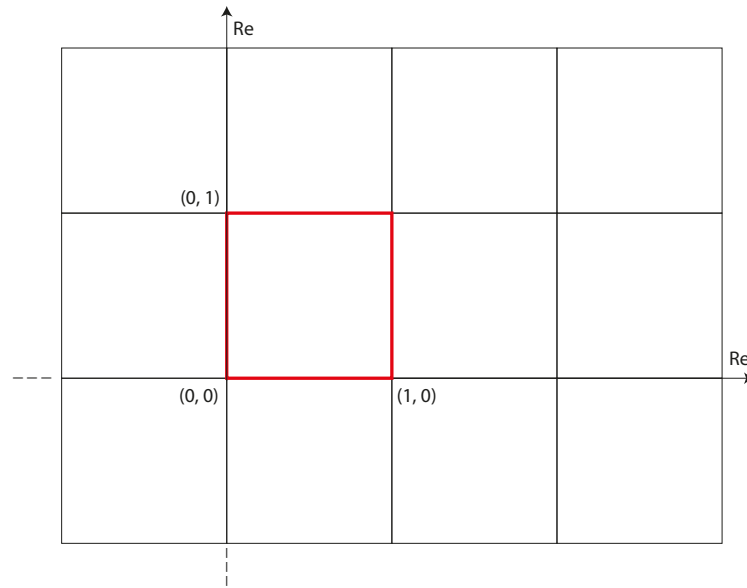


Figure 2.21: Fundamental domain for Example 2.25

Example 2.26 ($F_2 \times \mathbb{Z}$ acting on $T \times \mathbb{R}$). Finally, we shall consider a more involved example of a free group $F_2 \times \mathbb{Z}$ acting on a **tree** $\times \mathbb{R}$. We can see from Figure 2.22, how the actions on \mathbb{Z} result in moving along parallel trees living in \mathbb{R} . The fundamental domain for this example is given by $a \times b \times [0, 1]$, as highlighted in red in the illustration provided.

Definition 2.3.15 (Strict Fundamental Domain). A fundamental domain D is **strict** if it contains exactly one element for each orbit of G .¹⁸ This is equivalent to say that D intersects each orbit of G in exactly one point.

The difference between a fundamental domain and a strict fundamental domain is subtle, but it boils down to being able to define the smallest subset of a group describing the entire group. So a strict fundamental domain allows us to associate an optimal and concise subspace for our group G . We will present many examples of such domains in Chapter 4, where domains are used to create cellulations of topological spaces called Davis complexes.

¹⁸If two points in X are equivalent, then they belong to the same orbit. Thus, no two points in the fundamental domain are equivalent.

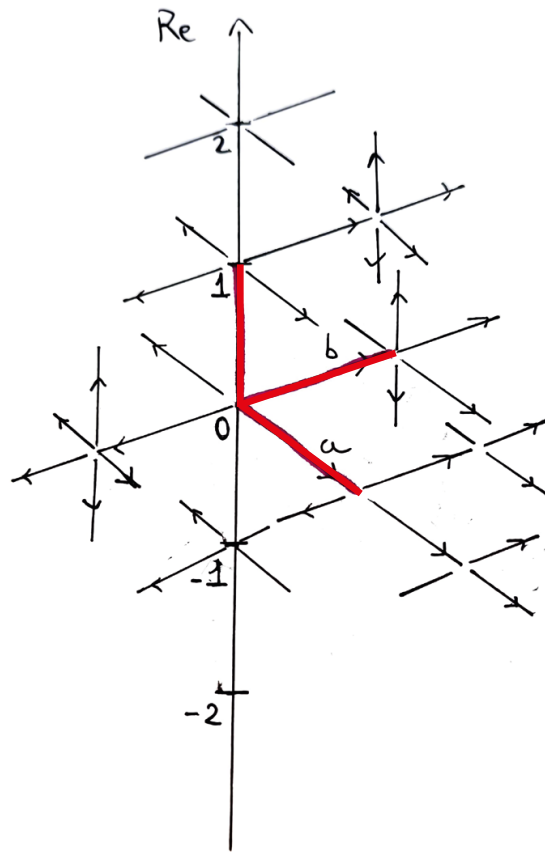


Figure 2.22: Example of how orbits determine the fundamental domain, highlighted in red.

Non Example 2.4. Examples 2.24 and 2.25 are not strict, because there is at least one point in their domain that are in the same orbit of another point. In the case of Example 2.24, the endpoints 0 and 1 belong to the same orbit. Similarly for Example 2.25.

Coxeter Groups

The content of this chapter, primarily based on [Sch17], will define Coxeter groups and provide some degree of in-depth understanding about them. In particular, we will explore how Coxeter groups manifest through Coxeter-Dynkin diagrams and how Cayley graphs change accordingly. The second part of this chapter provided a quick overview of the classification of finite Coxeter groups.

3.1 Groups and Systems

An important aspect of mathematics is the study of abstractions, or objects, provided some constraints. This is what allows mathematicians to study concepts and develop theories that can later be applied to many real world use cases. Likewise, we would like to study how objects reflect in space and we do so by looking at the family of groups generated by elements of order 2: **Coxeter groups**.

Definition 3.1.1 (Coxeter Group). A **Coxeter group** is a group satisfying the following conditions:

- All its generators $s_i \in S$ are of order 2, thus $s_i^2 = e$ with $i \in \{1, 2, \dots, n\}$.
- The combination of any two distinct generators s and s' has either a finite order $m \in \{2, 3, 4, \dots\}$ or infinite order $m = \infty$. We write: $(ss')^m = e$.

For a generator to have order 2, it means that it is its own inverse and therefore an involution. This property is equivalent to the geometric reflections we have seen in Section 2.2: a reflection of an object is itself. Similarly, we can think of generator pairs $(ss')^m$ as two “mirrors” such that an object requires m -reflections to be itself.

Definition 3.1.2 (Presentation of a Coxeter Group). Provided a set of generators S , with $s_i \in S$, the presentation of a Coxeter group is written as:

$$W = \langle s_1, s_2, \dots, s_n \mid s_1^2 = s_2^2 = \dots = s_n^2 = e, (s_i s_j)^{m_{ij}} = e \rangle$$

To further abbreviate the presentations we omit the identity and write:

$$W = \langle s_1, s_2, \dots, s_n \mid s_1^2, s_2^2, \dots, s_n^2, (s_i s_j)^{m_{ij}} \rangle$$

Example 3.1. The following is a first trivial example of a Coxeter group having one generator a :

$$G = \langle a \mid a^2 \rangle$$

This group accepts only two elements: $\{e, a\}$, thus making it isomorphic to the group of integers modulo 2:

$$G \cong \mathbb{Z}_2$$

Example 3.2. A Coxeter group with two generators, whose pair has order two, has representation:

$$G = \langle a, b \mid a^2, b^2, (ab)^2 \rangle$$

As we shall set below, $\langle a, b \mid a^2, b^2, (ab)^n \rangle \cong D_n$. Thus $G \cong D_2$ in this particular example.

Recall also that we can think of the generators as letters from an alphabet $S \cup S^{-1}$, such that we can form words under an *R-equivalence* relation (see Definition 2.1.13). Then G generates the words: $ab, abb, abba$, and so on. Further, we can use the relators from G to simplify words further, for example a word $UabababV$, can be reduced to $UabV$ since $abab = (ab)^2 = e$. Inverses of a word are obtained by flipping the order of its letters, for example $(ab)^{-1} = ba$, and reductions via R-equivalence work as just described.

Example 3.3. The Coxeter group with three generators and infinite order relationship between them has representation:

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^\infty, (bc)^\infty, (ac)^\infty \rangle$$

which can also write as:

$$G = \langle a, b, c \mid a^2, b^2, c^2 \rangle$$

G is the “freest” Coxeter group of three elements as it has an infinite number of elements generated by a, b , and c . If we think again in terms of words, the only reductions happen when a letter is adjacent to itself: $aa = bb = cc = e$. Otherwise, we can construct words of infinite length.

Non Example 3.1. The free group:

$$F_2 = \langle a, b \rangle$$

is not a Coxeter group, since its generators are not of order 2.

Non Example 3.2. In general, the free group:

$$F_n = \langle s, \dots, s_n \rangle$$

is not a Coxeter group, since none of its n-generators are of order 2.

Non Example 3.3. The group:

$$\langle a, b \mid a^2 \rangle$$

is not a Coxeter group, since one of its generators is not of order 2.

3.1.1 Coxeter-Dynkin Diagrams

We can represent a Coxeter group visually using a **Coxeter-Dynkin diagram** or **Coxeter diagram**.¹⁹ This is a mechanism to encode the group as a labelled graph. The rules to build one are simple:

- Each generator s_i is identified with a vertex.

¹⁹In this text may use the terms Coxeter diagram and Coxeter graph interchangeably. This is not to be confused with a Coxeter graph which is a non-Hamiltonian cubic symmetric graph having 28 vertices and 42 edges.[Weia]

- Each generator pair (s_i, s_j) is represented by a labelled edge with the associated order m_{ij} .
- Generators with infinite order, $m_{ij} = \infty$, are represented by the absence of an edge.

Where required, we will use the Greek *gamma* symbol Γ to indicate a Coxeter diagram.

We shall note that in some texts generator pairs of order two have no edge, those of order three have an unlabelled edge, whereas those of higher order are labelled ($m_{ij} \in \{4, 5, \dots\} \cup \infty$). However, as we will work with Coxeter groups with few generators we shall prefer labelling all edges of finite order. This has the advantage that all relations between generators are represented by labelled edges and not hidden to the reader.

Example 3.4. The Coxeter diagram for the Example 3.1 is given by:



Figure 3.1: Example of a trivial Coxeter diagram with 1 generator.

Example 3.5. The Coxeter diagram for the Example 3.2 is given by:

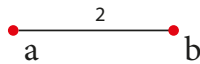


Figure 3.2: Example of a Coxeter diagram with 2 generators, whose pair has order two.

Example 3.6. The Coxeter diagram from Figure 3.3 has the Coxeter presentation:

$$G = \langle q, r, s, t \mid q^2, r^2, s^2, t^2, (sq)^2, (qt)^2, (sr)^3, (rt)^4 \rangle$$

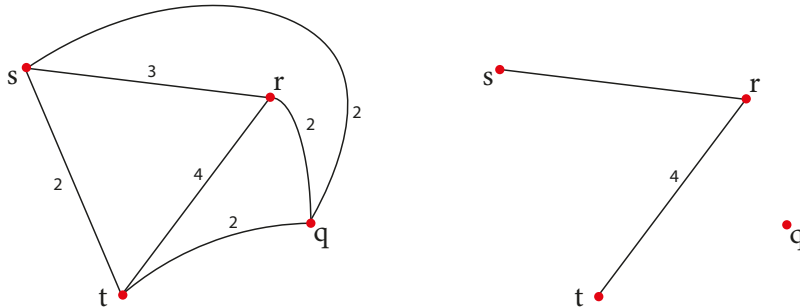


Figure 3.3: Example of a Coxeter-Dynkin diagram with 4 generators. The two diagrams are equivalent.

We note that group presentations and the corresponding Coxeter diagrams are equivalent and one can go from one to the other without loss of generality.

Example 3.7. Given a Coxeter graph $\Gamma = (V, E)$ for some group G , show that an automorphism of Γ preserving the edge labels induces an automorphism of the corresponding Coxeter group.

Proof. An automorphism on Γ is an isomorphism that “shuffles” the vertices while preserving adjacency and their labels. Thus there is a map $\phi : V_1 \rightarrow V_2$ between the vertices of two Coxeter graphs Γ_1 and Γ_2 . Since a Coxeter graph requires a bijection between edge labels and the order of generator pairs, then ϕ is isomorphic to the automorphism $\rho : G(\Gamma_1) \rightarrow G(\Gamma_2)$ on the corresponding Coxeter groups.

The automorphisms of a Coxeter graph Γ form a group $\text{Aut}(\Gamma)$. □

Remark 1. The automorphisms induced by the graph isomorphisms outlined above are particularly interesting as they are distinct from the trivial inner automorphisms arising from conjugations. The latter, in fact, act trivially on the group G by conjugation $ngx^{-1} \in G$, while the former must preserve the original structure of Γ and thus reflect the properties of a Coxeter group and its relations more clearly.

3.1.2 From Coxeter Diagrams to Cayley Graphs

In this section we recall the construction of Cayley graphs from Section 2.1.4 for some given group and compare how these differ from Coxeter diagrams.

Example 3.8. The Coxeter diagram and Cayley graph for the group:

$$G = \langle a, b \mid a^2, b^2, (ab)^2 \rangle$$

is shown in Figure 3.4. We note that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and also observe that the two generators commute, $ab = ba$. This is a property of any two generators with edge labelled 2, which we prove below.

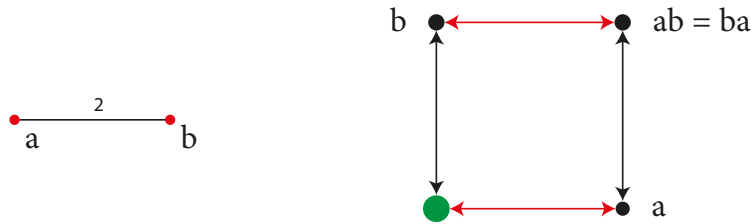


Figure 3.4: On the left the Coxeter diagram for Example 3.8 and on the right its Cayley graph.

Proof. Consider any two generators x and y of a Coxeter group such that $(xy)^2$. Then by definition:

$$\begin{aligned} xyxy &= e && \text{(multiply by } y) \\ xyxyy &= y && \text{(reduce } yy \text{ and multiply by } x) \\ xyxx &= yx && \text{(reduce } xx) \\ xy &= yx \end{aligned}$$

Thus any two generators with edge labelled 2 in a Coxeter diagram commute. □

Example 3.9. The Coxeter diagram and Cayley graph for the group:

$$G = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle$$

are shown in Figure 3.5. Note again that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and similarly to the previous example all generators commute.

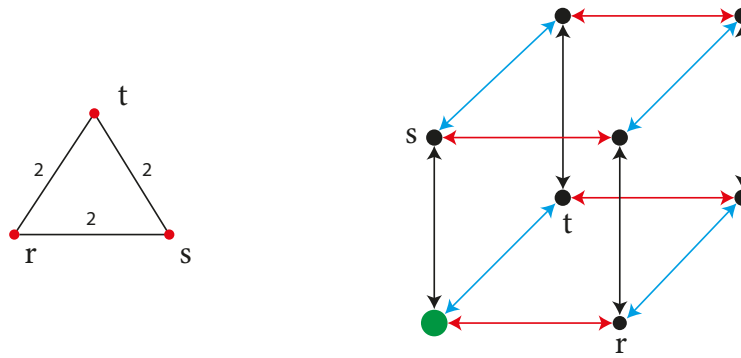


Figure 3.5: On the left the Coxeter diagram for Example 3.8 and on the right its Cayley graph.

Example 3.10. The Coxeter diagram and Cayley graph for the group:

$$G = \langle s_1, s_2, \dots, s_5 \mid s_1^2, s_2^2, \dots, s_5^2, (s_1 s_2)^2, \dots, (s_i s_{i+1})^2, \dots, (s_5 s_1)^2 \rangle$$

are shown in Figure 3.6. Unlike the two previous examples, not all generators commute with each other. We also observe that while we have drawn the complete Cayley graph identity (solid lines), the actual graphs extends infinitely (partial neighbourhood of the identity shown in dotted lines). Furthermore, we also observe that each vertex in the graph has five adjacent vertices, each representing the generators s_1, \dots, s_5 .

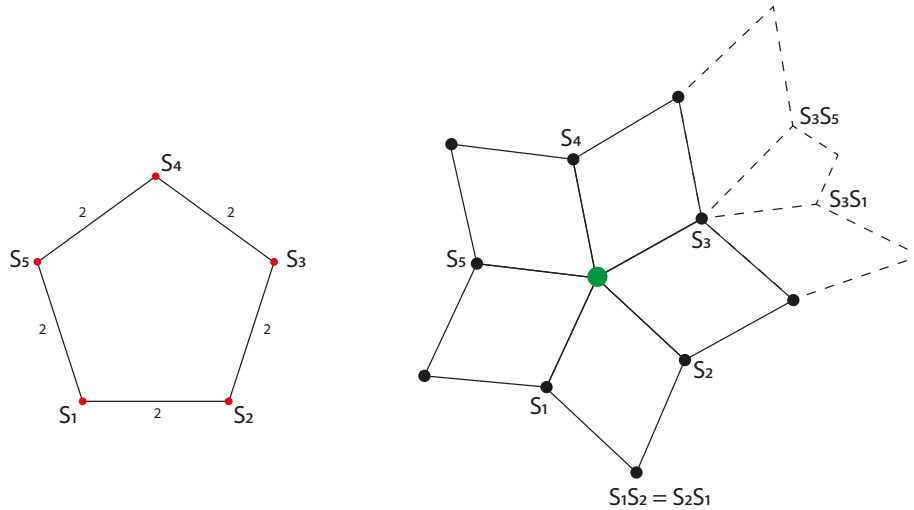


Figure 3.6: On the left the Coxeter diagram for Example 3.10 and on the right the identity of the Cayley graph (solid lines) and a partial neighbourhood (dotted lines). To simplify the Cayley diagram, the arrows are omitted and the same colour black is used for all generators.

Example 3.11. Finally, let's consider a modified version of the example above, such that one pair of generators has an order different from the others:

$$G' = \langle s_1, s_2, \dots, s_5 \mid s_1^2, s_2^2, \dots, s_5^2, (s_1 s_2)^2, \dots, (s_i s_{i+1})^2, \dots, (s_5 s_1)^3 \rangle$$

The Coxeter diagram and Cayley graph for G' are shown in Figure 3.7. It is worth noting that the Cayley graph remains invariant except for the edges representing the generators

that have a different order. Furthermore, the edges added in the Cayley graph are added in pairs, so that for any two generators $(s_i s_{i+1})^m$, we have $2m$ edges.²⁰

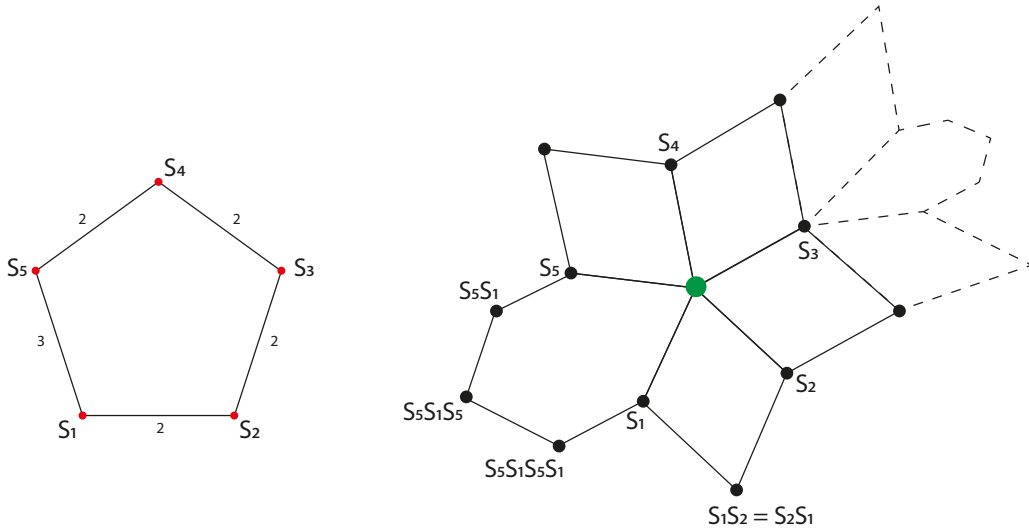


Figure 3.7: On the left the Coxeter diagram for Example 3.11 and on the right a partial Cayley graph.

We will explore reflections along generators of a group in the sections below of this chapter.

3.1.3 Geometric Interpretation of Coxeter Groups

Recall from Section 2.2 the geometric definition of reflections. Similarly, we can think of the generators of a Coxeter group as reflections, since each generator is also its own inverse.

If we construct an $n \times n$ matrix (a row and column for each generator) whose entries are the orders m_{ij} , then we obtain a **Coxeter matrix**.²¹ For the Example 3.6:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 2 & 4 & 2 & 1 \end{bmatrix}$$

This matrix represents an **adjacency matrix** for the Coxeter diagram. We also note that all diagonal entries are equal to 1 and since reflections are symmetric, we also have that any Coxeter matrix $A = A^T$, which is the required property of any symmetric matrix.

Example 3.12. The Coxeter matrices for Examples 3.8, 3.9, 3.10, and 3.11 are respectively:

²⁰It is understood that indices are all mod n , where n is the number of generators. In this particular example $n = 5$.

²¹It is important to note that m_{ij} is the order of the product of **two** generators. Thus $(s_i s_i)$ is always of order 1 since $s_i^2 = (s_i s_i)^1 = e$ in any Coxeter group.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & \infty & \infty & 2 \\ 2 & 1 & 2 & \infty & \infty \\ \infty & 2 & 1 & 2 & \infty \\ \infty & \infty & 2 & 1 & 2 \\ 2 & \infty & \infty & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & \infty & \infty & 3 \\ 2 & 1 & 2 & \infty & \infty \\ \infty & 2 & 1 & 2 & \infty \\ \infty & \infty & 2 & 1 & 2 \\ 3 & \infty & \infty & 2 & 1 \end{bmatrix}$$

Example 3.13. Consider the 2-dimensional Euclidean geometric model. Find a presentation for the group generated by a reflection over the x -axis and the line $y = \sqrt{3}x$.

Solution by geometric construction:

First of all, we plot the two lines $x = 0$ and $y = \sqrt{3}x$ on the Cartesian plane and observe that their angle is $\alpha = \sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$. Then we extend the original lines and their reflections so that we have a Coxeter group on the Euclidean plane, as shown in Figure 3.8. We also observe reflections along the generators, the coloured dots in the figure, which one may think of as reflections on two mirrors placed at a 60° angle with each other.

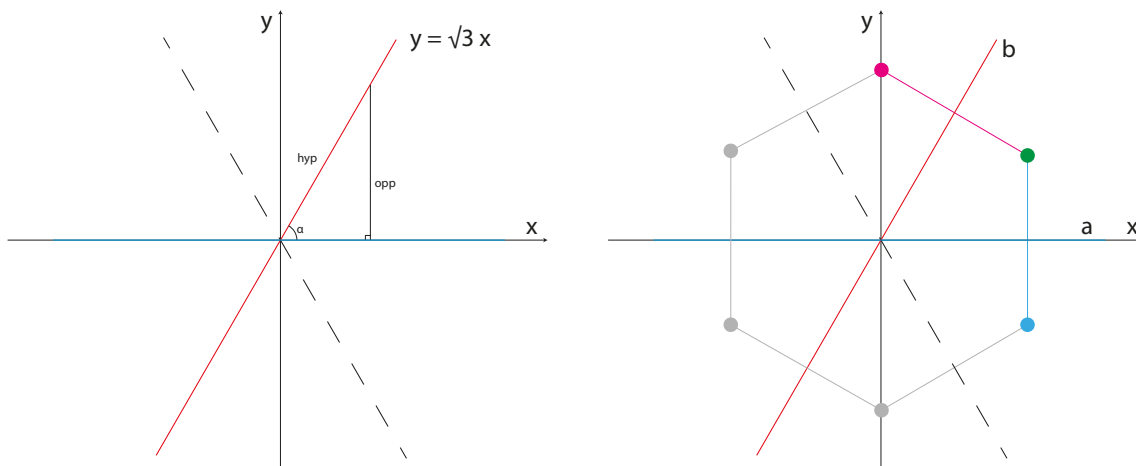


Figure 3.8: On the left the x and $y = \sqrt{3}x$ lines are drawn on the Cartesian plane. On the right, the corresponding generators, a and b , are shown with their reflections. The green dot is the identity, the blue one is the reflection along the generator a , the magenta one is the reflection along the generator b , and so on.

Thus we conclude that the presentation is:

$$\langle a, b \mid a^2, b^2, (ab)^3 \rangle$$

This has Coxeter diagram and Cayley graph as shown in Figure 3.9. We also note that this is the Dihedral group of order 6: $D_3 \cong S_3$ and recall from Example 2.10 that all Dihedral groups are Coxeter groups. D_3 is the dihedral group of reflections and rotations for an equilateral triangle, see Figure 3.10.

Similarly, we can find reflections and Coxeter groups for reflections along any arbitrary lines in space. In particular, if any two lines are at an angle of $\frac{2\pi}{n}$, then we have a Coxeter group for D_n . However, in some cases this can be challenging for other angles or if the

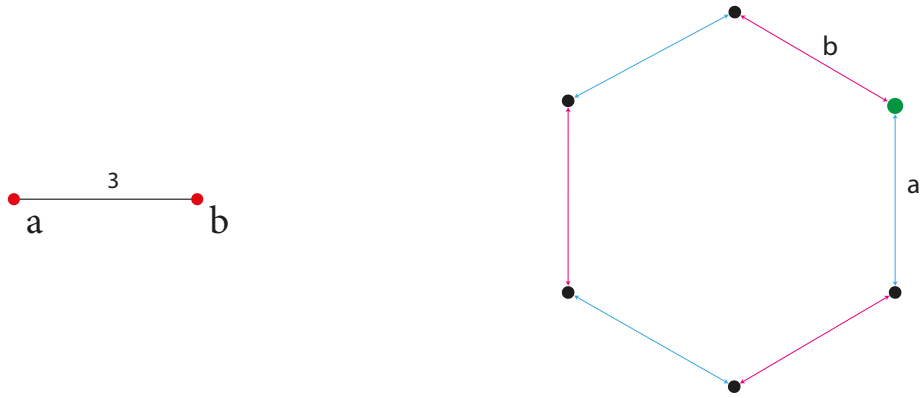


Figure 3.9: On the left the Coxeter diagram for $\langle a, b \mid a^2, b^2, (ab)^3 \rangle$ and on the right its corresponding Cayley graph.

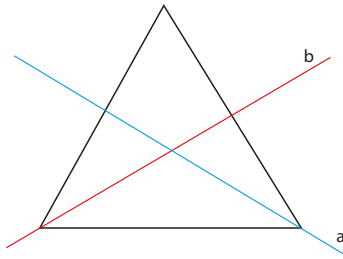


Figure 3.10: Equilateral triangle representing D_3 . The two generators a and b are the lines of reflections.

number of lines of reflections increases. Especially for Coxeter groups that live on non-Euclidean space, which means that any of its visual representations requires some form of deformation when drawn on paper.

Definition 3.1.3 (Coxeter System). The tuple (W, S) , where W is a Coxeter group and S a set of generators, is called a **Coxeter system**.

Example 3.14. The standard Coxeter presentation for D_6 is given by:

$$\langle a, b \mid a^2, b^2, (ab)^6 \rangle$$

Thus we have a Coxeter system $(D_6, \{a, b\})$. However, the group can also have presentation:

$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^2, (ac)^3 \rangle$$

Thus we have a isomorphic Coxeter system $(D_6, \{a, b, c\})$, where $|S| \neq |S'|$.

The number of generators $|S|$ is called the **rank** of the Coxeter system. We also note that the rank corresponds to the size n for the corresponding $n \times n$ Coxeter matrix. Whenever the set of generators is understood, we use the terms Coxeter system and group interchangeably.

Example 3.15. The Coxeter system for Example 3.13 is $(G, \{a, b\})$ and has rank 2.

Example 3.16. The Coxeter system for Example 3.9 is $(G, \{r, s, t\})$ and has rank 3.

Example 3.17. The Coxeter systems for Examples 3.10 and 3.11 are $(G, \{s_1, \dots, s_5\})$ and $(G', \{s_1, \dots, s_5\})$ respectively, both with rank 5.

Definition 3.1.4 (Finite Coxeter Group). A Coxeter group is **finite** if the number of elements in the group is finite.

We may think of a finite Coxeter group as one that generates reflections in a tiling pattern that is contained in the space, rather than extending indefinitely. For instance, the Coxeter group from the Example 4.2 is equivalent to the infinite tessellation of equilateral triangles in the Euclidean space, hence it is not a finite Coxeter group.

3.2 Classification of Finite Coxeter Groups

Two seminal papers of Coxeter from 1934 and 1935 classify all finite groups using the irreducibility of groups.[Cox34, Cox35] This section does not go into the details of these papers, but given their importance in the subject we do list its primary results.

3.2.1 The Schläfli Matrix

Definition 3.2.1 (Schläfli Matrix). Provided a Coxeter matrix M , the **Schläfli matrix** is given by:

$$A = -2 \cos\left(\frac{\pi}{M_{ij}}\right)$$

Solving the characteristic polynomial given by:

$$\det(A - \lambda I) = 0$$

we find the eigenvalues of A , which can be used to classify Coxeter groups into:

- **Finite:** all eigenvalues are positive (**positive definite**).
- **Affine:** all eigenvalues are non-negative, and at least one is zero (**positive semi-definite**).
- **Indefinite:** if not finite or affine.[Hum90, BB10]

Equivalently, from linear algebra, we know that a matrix A is *positive definite* if and only if all its leading principal minors (LPMs) are positive.²² Thus, a Coxeter group is finite if and only if all its LPMs are positive.

Example 3.18. The Schläfli matrix for the Coxeter group from Example 3.8 is given by:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

This has eigenvalue 2 with algebraic multiplicity 2. Being the eigenvalue positive, the Coxeter group is finite.

²²A **minor** of A is the determinant of a square submatrix of A . If the submatrix is an upper-left submatrix of A , then it is called a **leading principal minor**. Refer to any standard linear algebra textbook for a more precise definition.

Example 3.19 (Schläfli Matrix for D_3). The Coxeter matrix for D_3 is:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

and its corresponding Schläfli matrix is:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

This has eigenvalues 3 and 1. Both the eigenvalues are positive, hence the group is finite.

Example 3.20. Let's consider the Coxeter group from Example 3.9:

$$G = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle$$

This has a Schläfli matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus its eigenvalue 2 (with algebraic multiplicity 3) is positive and the group is finite.

3.2.2 The Finite Reflection Groups

For this section, we shall use the construction of Coxeter-Dynkin diagram where generator pairs of order two have no edge, those of order three have an unlabelled edge, and those of higher order are labelled $(m_{ij} \in \{4, 5, \dots\} \cup \infty)$. Then, Coxeter groups can be classified as irreducible or reducible as per the following definitions.

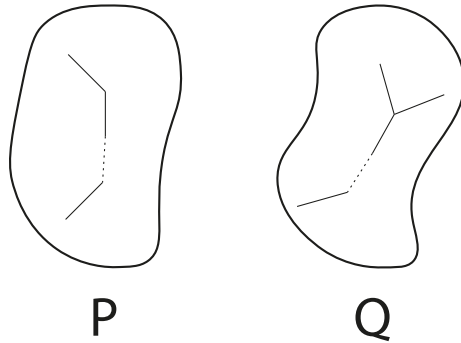


Figure 3.11: Abstract representation of a reducible group $G \cong P \times Q$.

Definition 3.2.2 (Irreducible). A Coxeter group is **irreducible** if its Coxeter graph is connected.

Definition 3.2.3 (Reducible). A Coxeter group is **reducible** if it is not *irreducible*. A reducible group G can also be thought of as the direct product $P \times Q$ of two irreducible groups P and Q .

Figure 3.11 illustrates, abstractly, how two irreducible groups can be combined to create a new group. The fact that we have no edges between the two groups P and Q means that each vertex pair (p, q) , where $p \in P$ and $q \in Q$, has order 2.²³ Having all such pairs order 2, P and Q commute, hence $P \times Q \cong Q \times P$.

The distinction between reducible and irreducible groups allows us to factorise groups in terms of primes, similarly to how primes work for the natural numbers. Thus one can construct any finite Coxeter group using a well-known dictionary of irreducible Coxeter groups. This dictionary is shown in Figure 3.12 and from it one can show that each of these corresponds to a finite Coxeter group.[Hum90] Given that each finite Coxeter group is either irreducible or the combination of other irreducible groups, the problem reduces to calculating $\det(A)$ for the graphs of the corresponding irreducible groups, which are the ones from the table provided below, and noting that since $\det(A) > 0$, the LPMs are all positives and therefore all groups are finite.

$n + 1$	2	$4 \sin^2(\pi/k)$	$3 - \sqrt{5}$	1	$\frac{7-3\sqrt{5}}{2}$	4	3	2	1
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Table 3.1: Table of $\det(A)$ for each finite Coxeter group, as shown in Figure 3.12 in order.

²³If we were to draw such edges, we would obtain a complete bipartite graph $K_{P,Q}$.

Symbol	Graph	Order*
$[3^{n-1}] \dagger \quad (n \geq 1)$		$(n+1)!$
$[3^{m-2}, 4] \quad (m \geq 2)$		$2^m m!$
$[k] \ddagger \quad (k \geq 5)$		$2k$
$[3, 5] \S$		120
$[3, 4, 3]$		1152
$[3, 3, 5]$		14400
$\begin{bmatrix} 3^{m-3} \\ 3 \\ 3 \end{bmatrix} \parallel \quad (m \geq 4)$		$2^{m-1} m!$
$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix} \parallel$		51840
$\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$		2903040
$\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$		696729600

Figure 3.12: List of Coxeter-Dynkin diagrams for the full enumeration of finite groups. Image from [Cox35]. The graphs have a missing edge for $m_{ij} = 2$, unlabelled edges for $m_{ij} = 3$, and all other edges are labelled. The square bracket notation used by Coxeter is explained in his earlier paper [Cox34].

Coxeter Cellulations

A **Davis complex** Σ is a topological complex associated to a Coxeter system (W, S) , where the group W acts both *properly* and *co-compactly* (see Definitions 2.1.18 and 2.3.13 respectively). There are multiple ways to build a Davis complex, among which: as the quotient space of the group with respect to s-mirrors and as a CW-complex. In this chapter, we will explore these constructions in order to gain a better understanding of Coxeter systems. Examples will be provided for different cellulations in Euclidean and non-Euclidean space.

4.1 Chambers and Mirror Cellulations

In order to build a Davis complex and its cellulation, we need to start from the ground up. We begin with the definition of spherical subsets and look at Coxeter groups of rank 3. We will then use this to define chambers for our Coxeter systems and finally we will look at mirror cellulations for Davis complexes.

Definition 4.1.1 (Spherical Subset). Given a set of generators S for some Coxeter system (W, S) , subsets $U \subseteq S$ of generators that generate finite groups (*i.e.*, there are a finite number of elements in the group) are called **spherical subsets**. For groups generated by U we use the notation W_U .

The number of generators in a spherical subset is called the **order** of the subset.

For the purpose of this section, we will focus on spherical subsets of dimension 3 and outline an important result about order of generators and whether a group is finite or not in the theorem below.

Theorem 4.1.1. *The Coxeter group generated by three generators $\{r, s, t\}$ is finite if and only if:*

$$\frac{1}{m_{rs}} + \frac{1}{m_{st}} + \frac{1}{m_{rt}} > 1$$

Groups satisfying this property are called **spherical triangle groups** and it is trivial to see how there are only four possible triplets satisfying this condition:

- (2, 3, 3): this is the symmetry group of the tetrahedron.
- (2, 3, 4): this is the symmetry group of the cube.
- (2, 3, 5): this is the symmetry group of the dodecahedron.
- (2, 2, n), where $n \geq 2$: this is the group $C_2 \times D_n$.

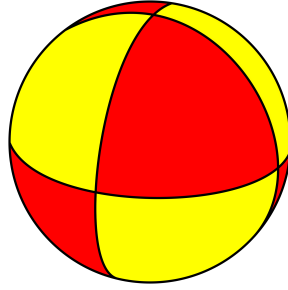


Figure 4.1: Spherical tessellation for the Coxeter triangle group $(2, 2, 2)$.

The inequality can be “modified” for the Euclidean ($= 1$) and hyperbolic (< 1) cases, as outlined by [Dav12, Chapter 6]. Being these triangle groups, we can associate an angle π/m_{ij} to each generator pair and recall that only in the Euclidean space the sum of the angles of a triangle is π , hence:

$$\frac{\pi}{m_{rs}} + \frac{\pi}{m_{st}} + \frac{\pi}{m_{rt}} = 1$$

while in the hyperbolic and spherical spaces, the sum is less or greater than one respectively. A complete proof of this theorem is involved, requiring to consider all three cases ($=, <, >$) and proving the implications in both directions (\implies and \impliedby), thus it will be omitted from the scope of this dissertation.

Example 4.1. Let’s consider the Coxeter group from Example 3.9, which we have already shown to be finite via the Schläfli matrix in the previous chapter. In this case, $m_{rs} = m_{rt} = m_{st} = 2$, hence:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} > 1$$

Thus the group is finite and its tessellation is shown in Figure 4.1.

We should also note that each vertex (*i.e.*, generator) corresponds to a spherical subset of order 1. Similarly, the generator pairs, represented by edges in the corresponding Coxeter diagram, correspond to spherical subsets of order 2.

Example 4.2. Let’s consider the Coxeter group:

$$G = \langle r, s, t \mid r^2, s^2, t^2, (rs)^3, (rt)^3, (st)^3 \rangle$$

In this case, $m_{rs} = m_{rt} = m_{st} = 3$, hence:

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Thus the group is not finite.

We can double check via the Schläfli matrix:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

This has eigenvalues 3 (algebraic multiplicity 2) and 0, thus the group is *affine*.

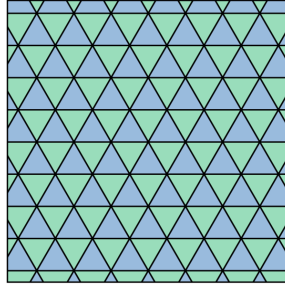


Figure 4.2: Euclidean tessellation for the Coxeter triangle group with generator pairs of order $(3, 3, 3)$.

It is easy to show through a simple geometric process that this group generates an infinite group in the 2-dimensional Euclidean space of triangle tiles, as shown in Figure 4.2, which can extend indefinitely on the plane.

Similarly to the finite case (> 1), there are a finite number of such groups: $(3, 3, 3)$, $(4, 4, 2)$, and $(6, 3, 2)$. This can be proven trivially by looking at all possible combinations of three positive integer numbers such that the sum of their reciprocals is one.

Example 4.3. Let's consider the Coxeter group:

$$G = \langle r, s, t \mid r^2, s^2, t^2, (rs)^3, (rt)^3, (st)^4 \rangle$$

In this case, the sum of reciprocal orders is:

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{11}{12} < 1$$

Hence, this is the hyperbolic case and the group is not finite. There are an infinite number of such groups, some of which were of inspiration to Escher when working on the Circle Limit woodcuts. Figure 4.3 is a representation of the group $(3, 3, 4)$ on a Poincaré disc and reminiscent of Escher's work.²⁴

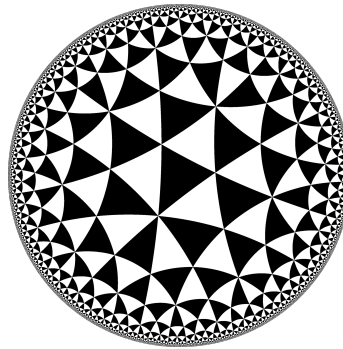


Figure 4.3: Hyperbolic tessellation on the Poincaré disc for the Coxeter triangle group with generator pairs of order $(3, 3, 4)$.

²⁴Note that in Escher's work, motifs rather than regular polygons are reflected across the space. For instance, in *Circle Limit IV*, the motif is a bat.

4.1.1 The Strict Fundamental Domain K

The strict fundamental domain K for a given Coxeter system (W, S) , also known as **chamber**²⁵, it is used to build the Davis complex (see Definition 2.3.15). First of all, we shall consider the poset for the generators S such that they are all spherical subsets and with order \subseteq . Then we associate spherical subsets to n -cells as such:

- All spherical subsets $(\emptyset, \{s_1\}, \{s_2\}, \dots, \{s_1, s_2, \dots, s_n\})$ are mapped to 0-cells (vertices).
- For any two spherical subsets U and V such that $U \subseteq V$, we associate a 1-cell (edge) with vertices being the 0-cells for U and V .
- Given three spherical subsets U, V , and W such that $U \subseteq V \subseteq W$, we associate a 2-cell (face) with edges being the 1-cells from $U \subseteq V, V \subseteq W$, and $U \subseteq W$.
- The process continues for n -order posets associated to $(n - 1)$ -cells.

This construction of K requires the gluing of n -dimensional triangles (*i.e.*, simplexes), making it a simplicial complex (see Definition 2.3.7). For this reason, we say that K is the **geometric realisation** of the posets of spherical subsets.²⁶

Example 4.4. Let's consider again the Coxeter group from Example 3.13:

$$G = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$$

Its spherical subsets are: $\emptyset, \{a\}, \{b\}, \{a, b\}$.

The creation of the strict fundamental domain K is shown in Figure 4.4.

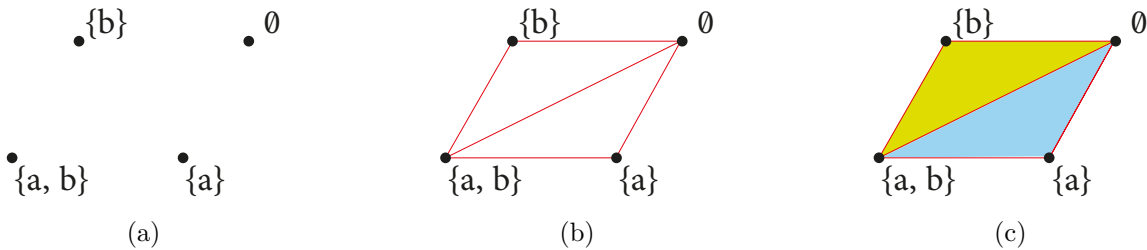


Figure 4.4: Step-by-step process for the creation of the strict fundamental domain K for D_3 . (a) shows 0-cells added as vertices, one for each set in the poset. (b) shows 1-cells are added as edges, one for each pair in the poset. Finally, (c) shows 2-cells are added as triangular faces for each triplet in the poset.

4.1.2 The Davis Complex as a Quotient Space

Definition 4.1.2 (Direct Product of Group and Domain). Given a group G and a domain D , their direct product is defined as:

$$G \times D = \{g \times D \mid g \in G\}$$

²⁵We will use the terms chamber, strict fundamental domain, and domain interchangeably.

²⁶Note the resemblance with Hasse diagrams of posets. See Appendix A.

where:

$$g \times D = \{(g, d) \mid d \in D\}$$

We note, that this is equivalent to a left-group action, hence we can equivalently write:

$$G \times D = \{gD \mid g \in G\}$$

Definition 4.1.3 (s-mirror). The set of cells of K containing the vertex s is called the **s-mirror**, upon the condition that s corresponds to a generator for the Coxeter group, $s \in S$.

Definition 4.1.4 (Davis Complex). We define the Davis complex as the quotient space of the group acting on the strict fundamental domain K :

$$\Sigma = (W \times K)/\sim$$

Where the equivalence relation for some s-mirror is defined as:

$$\sim: W \times K$$

such that:

$$(w, x) \sim (v, y) \iff x = y, v = ws$$

Geometrically, we can think of this definition of the Davis complex as reflecting combinatorially the domain K across all s-mirrors.

Example 4.5. Let's consider again the Coxeter group D_3 from Example 3.13. In this case, the strict fundamental domain K is defined as vertices from the spherical subsets \emptyset , $\{a\}$, $\{b\}$, and $\{a, b\}$. Reflecting the fundamental domain K by the s-mirrors b , a , ba , and ab , we obtain the hexagonal complex shown in Figure 4.5.

Proposition 4.1.1. *The reflection relation \sim on $W \times K$ generates an equivalence relation.*

Proof. To prove this proposition we need to verify the three axioms of the equivalence relation \sim (see Definition 2.1.10):

- **Reflexivity:** $(w, x) \sim (w, x)$. Since $x = x$ and $w = we$,²⁷ then $(w, x) \sim (w, x)$ holds.
- **Symmetry:** if $(w, x) \sim (v, y)$ then $(v, y) \sim (w, x)$. First of all, we start with the assumption that $(w, x) \sim (v, y)$ holds and thus $x = y$ and $v = ws$ by definition. Since $x = y$, then $y = x$. Then assume there exists a generator s' such that $w = vs'$. Then $w = vs' = (ws)s' = w$, provided that $s' = s^{-1}$. Thus the symmetry condition holds.
- **Transitivity:** if $(w, x) \sim (v, y)$ and $(v, y) \sim (u, z)$, then $(w, x) \sim (u, z)$. By assumption, we have that $x = y$ and $y = z$, then it follows by equality that $x = z$. We also know that $v = ws$ and $u = vt$ for two generators s and t . Then $u = (ws)t = w(st) = ws'$ for some generator $s' \in S$. Thus the transitivity condition holds.

All three axioms are satisfied, hence \sim is an equivalence relation. □

²⁷ e is the s-mirror along the identity generator. Thus any reflection along e is itself.

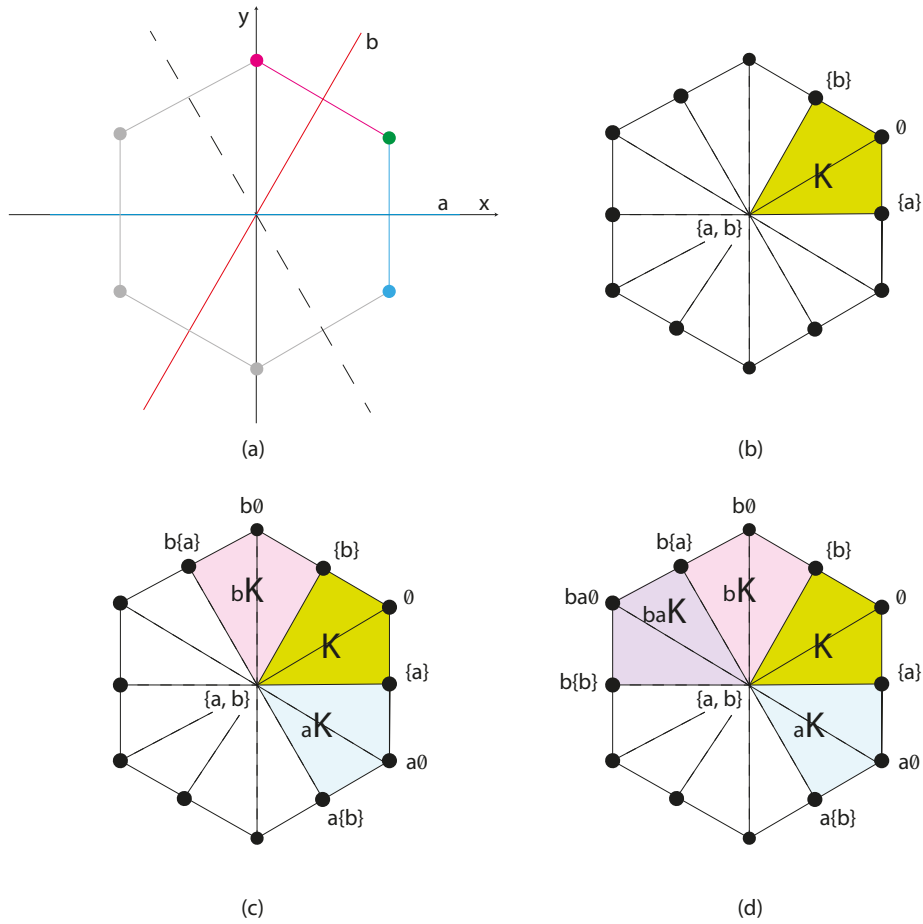


Figure 4.5: This illustration shows (a) the Cayley diagram for the Coxeter group D_3 of Example 4.5 and (b, c, d) some of the construction steps for the Davis complex as generated via s-mirrors. The strict fundamental domain K is shaded in (b) and reflected along the b -mirror (with edge $\{b\}$ — $\{a, b\}$) and the a -mirror (with edge $\{a\}$ — $\{a, b\}$) in (c). The reflection bK is further mirrored along the a -mirror to obtain baK . Further reflections along the composed mirrors complete the Davis complex.

Proposition 4.1.2. *If $(v, x) \sim (u, y)$, then for any $w \in W$, we have that $w \cdot (v, x) \sim w \cdot (u, y)$*

Proof. The left action is defined as $w \cdot (v, k) = (wk, k)$. Then in order to show this proposition, we shall apply the left action and show that the equivalence relation axioms still hold.

- **Reflexivity:** $w \cdot (v, x) \sim w \cdot (v, x)$. Since $x = x$ and $w \cdot v = wk = wve$, then the reflexivity condition holds.
- **Symmetry:** if $w \cdot (v, x) \sim w \cdot (u, y)$ then $w \cdot (u, y) \sim w \cdot (v, x)$. By assumption we have that $x = y$ and thus $y = x$. For the generators, we have that $w \cdot u = wk = wu = w(vs) = w \cdot vs = w \cdot u$, since $vs = u$ from our previous proof. Thus the symmetry condition holds.

- **Transitivity:** if $w \cdot (v, x) \sim w \cdot (u, y)$ and $w \cdot (u, y) \sim w \cdot (q, z)$, then $w \cdot (v, x) \sim w \cdot (q, z)$. By assumption, we have that $x = y$ and $y = z$, then it follows by equality that $x = z$. Then we have that $w \cdot u = wu = w(vs)$ and $w \cdot q = wq = w(us)$. The generator q can be rewritten as vs' , as shown in the previous proof, and therefore $w \cdot v = w \cdot q$. Thus the transitivity condition holds.

□

Proposition 4.1.3. *The action $w \cdot (v, x) = (wv, x)$ is cellular, proper, co-compact, and has chamber K .*

Proof. We prove each assertion separately.

Cellular Action: given $w \in W$ and (v, x) associating a group element to an element in the cell corresponding to the chamber K , then the group element $wv \in W$. Hence the action sends cells to cells and thus it is cellular.

Proper Action: the stabilizers of the cells in Σ are those actions sending cells to themselves. By definition these are the identity element in W and all the Coxeter reflections: s_i^2 and $(s_i s_j)^{m_{ij}}$. Given that we have a finite number of relations, $|\text{Stab}_W| < \infty$ and therefore the action is proper.

Co-compact Action: given that the set of generators S is finite, K is also a finite simplicial complex and compact. Hence, the quotient space $(W \times K)/\sim$ is a finite complex and the action acting on it is co-compact by definition.

Chamber K : given a chamber $K \in \Sigma$, the action sends each point in K to another cell in the complex. The action acts as an equivalence relation (see Proposition 4.1.2), hence we can identify the action with a chamber K . □

For the case where the Coxeter graph is an n -gon, it can be challenging to visualise the strict fundamental domain K and its reflections. However, the cellulation can be simplified further by having a simpler complex and looking only at the s -mirrors for K . We do so by removing the 0-cells for the single elements spherical subsets and keeping only edges between order-1 spherical subsets and order-2 spherical subsets, with inclusion \subseteq as our poset order.

Example 4.6. The Coxeter group:

$$G = \langle r, s, t \mid r^2, s^2, t^2, (st)^2, (sr)^2, (rt)^2 \rangle$$

is equivalent to the Coxeter diagram from Figure 4.6a. This requires a Cayley graph, having the 1-skeleton of a 3-cube, as shown in Example 3.9 while its Davis complex Σ can be visualised via mirror cellulations as in Figure 4.6b. We recall also from earlier discussions of this group in Examples 3.20 and 4.1 that this is a finite group and the tessellation of the domain K lives in a spherical space. Hence, with the help of some imagination the diagram for this group should be wrapped around a 2-sphere. Indeed, note that an equilateral triangle in the Euclidean space requires its internal angles to be 60° each, but when tiling

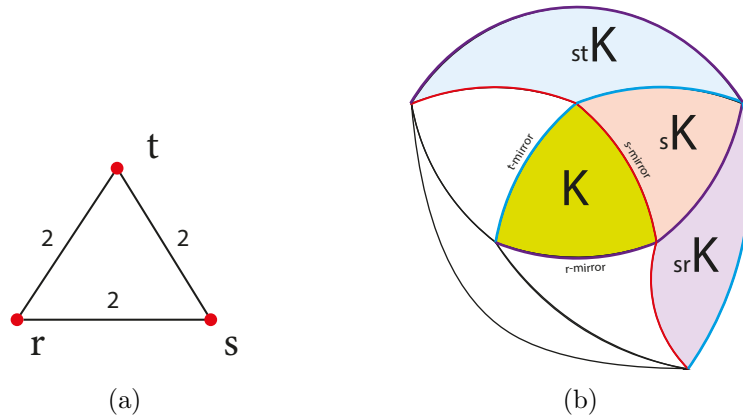


Figure 4.6: (a) The Coxeter diagram for $(2, 2, 2)$. (b) The mirror cellulations for the Davis complex of the Coxeter group $(2, 2, 2)$. The green shaded triangle is the fundamental domain K and its edges are the three s -mirrors corresponding to the three generators. This diagram illustrates only some of the mirrored domains.

K we required four adjacent such triangles, thus each angle needs to be 90° . This is only allowed in the spherical space.

Example 4.7. The Davis complex Σ for the Coxeter group:

$$G = \langle r, s, t \mid r^2, s^2, t^2, (st)^2, (sr)^4, (rt)^4 \rangle$$

has diagram 4.7a and mirror cellulation 4.7b. From Theorem 4.1.1 we know that since the sum of the reciprocals of the pair orders is one, then the group is infinite in the Euclidean space.²⁸ Unlike the previous example, we also note that since the order of the generator pairs differ, we can accept a degree of asymmetry in the angles for the triangle complex K . Thus K does not need to be equilateral and we have angles 90° for $(rs)^2$ and 45° for $(rt)^4$ and $(st)^4$. In fact, this generalises so that we have angle $\frac{\pi}{m_{ij}}$ for the generators s_i and s_j .

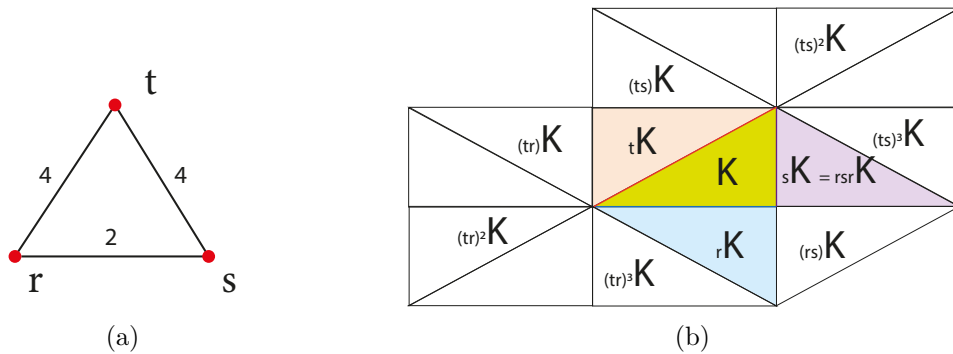


Figure 4.7: (a) The Coxeter diagram for $(2, 4, 4)$. (b) Partial mirror cellulations for the Davis complex of the Coxeter group $(2, 4, 4)$. The green shaded triangle is the fundamental domain K and its edges are the three s -mirrors corresponding to the three generators. This diagram illustrates only some of the mirrored domains.

²⁸This is also known as the Tetrakis square tiling.

We should note that finite groups generate a finite number of words and this is clearly visible from the tessellation of Example 4.6, while groups with cellulations in the Euclidean space accept words of infinite length. In fact, a word can be seen as applying a reflection through an s -mirror, which can be done indefinitely in \mathbb{E}^n and \mathbb{H}^n (see Example 4.7).

Example 4.8. The infinite Dihedral group $D_\infty = \langle r, s \mid r^2, s^2 \rangle$ has the Coxeter group as shown in Figure 4.8a and the mirror cellulation of Σ is as shown in Figure 4.8b. The chamber K is the segment between $\{r\}$ and $\{s\}$, each endpoint corresponding to an s -mirror. Cellulations result from mirroring K along each s -mirror.

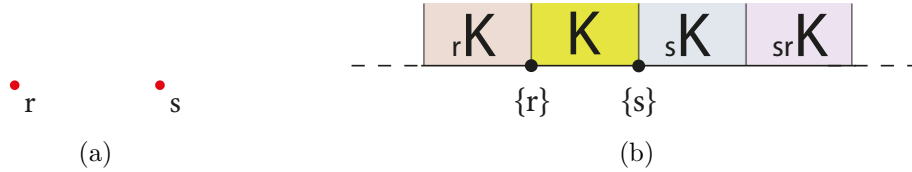


Figure 4.8: (a) The Coxeter diagram for D_∞ . (b) Mirror cellulations for the Davis complex of the Coxeter group D_∞ . The segment highlighted by green shaded area is the fundamental domain K and its vertices are the s -mirrors corresponding to the two generators r and s .

4.2 Coxeter Cellulation Σ_{cc} of the Davis Complex

The construction of a Davis complex via posets and mirror cellulations requires many steps in order to obtain a valid simplicial complex. However, we can obtain an equivalent construction by building it as a CW-complex. This is called **Coxeter cellulation** and it is denoted as Σ_{cc} .

The construction of the chamber is similar to the geometrical realisation of posets, except that the resulting complex is CW and the cells required at each step are topological balls instead of simplices. We call these **Coxeter cells of type T** and we construct them starting from the spherical subsets of the group generators:

- The empty set \emptyset is a cell corresponding to a point in the space ($T = \emptyset$, 0-cell).
- The spherical sets with one elements are cells corresponding to edges in the space ($T = \{s\}$, 1-cell).
- The spherical sets with two elements are cells corresponding to polygons with $2m_{st}$ sides in the space ($T = \{s, t\}$, 2-cell).
- And so on for n -balls in higher dimensions.

Since we build the T -cells from spherical subsets, each will correspond to a coset W_T for the Coxeter group W .

Example 4.9. Let's consider again the Coxeter group isomorphic to D_3 :

$$G = \langle r, s \mid r^2, s^2, (rs)^3 \rangle$$

Its Coxeter cellulations is built as following. First, we determine its spherical subsets:

- \emptyset corresponding to a 0-cell.

- $\{r\}$ and $\{s\}$, each corresponding to a 1-cell. Hence, we have two kinds of edges in our complex.
- $\{r, s\}$ corresponding to a 2-cell.

We then look at all cosets for each T-cell:

- W_\emptyset has cosets: $W_\emptyset, rW_\emptyset, (rs)W_\emptyset, (rs)rW_\emptyset, (rs)^2W_\emptyset, (rs)^2rW_\emptyset$.²⁹ Each coset is identified with a vertex.
- $W_{\{r\}}$ has cosets: $W_{\{r\}}, sW_{\{r\}},$ and $(rs)W_{\{r\}},$ each corresponding to edges with vertices: $(\emptyset, r), (s, sr),$ and (rs, rsr) .³⁰
- $W_{\{s\}}$ has cosets: $W_{\{s\}}, rW_{\{s\}},$ and $(sr)W_{\{s\}},$ each corresponding to an edge with vertices: $(\emptyset, s), (r, rs),$ and (sr, srs) .
- $W_{\{r,s\}}$ has one coset of six elements, itself: $\{\emptyset, r, (rs), (rs)r, (rs)^2, (rs)^2r\}$. These elements correspond to the hexagon with vertices and edges defined by the elements in $W_\emptyset, W_{\{r\}},$ and $W_{\{s\}}.$

Gluing vertices, edges and faces together we form the Coxeter cellulation Σ_{cc} for G , which is a CW-complex, as shown in Figure 4.9. Recall that $G \cong D_3$ and note that there is a 1:1 correspondence with its Cayley diagram (see Figure 3.9). Furthermore, comparing Σ_{cc} for D_3 with the cellulation obtained from s-mirroring in Example 4.5, we note how the latter requires vertices and edges that become unnecessary in Σ_{cc} . This is possible because we allow our Davis complex to be a CW-complex. However, in doing so we lose the details of the chamber of the group.

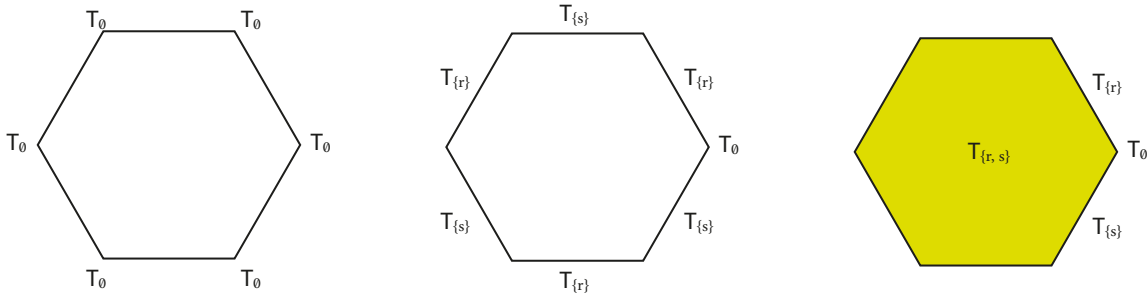


Figure 4.9: Coxeter cellulation for the group of Example 4.9. The process for cells at each dimension is illustrated from left-to-right.

Example 4.10. The Coxeter cellulation Σ_{cc} for the spherical triangular group:

$$G = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (st)^2, (rt)^2 \rangle$$

is obtained as such. First of all, we declare all its spherical subsets:

- \emptyset corresponding to a 0-cell.

²⁹Note that $(rs)^2r = s, (rs)^2 = (sr),$ etc...

³⁰Note that $sW_{\{r\}} = (rs)^2W_{\{r\}}$ corresponding to edge with vertices $((rs)^2, (rs)^2r)$.

- $\{r\}$, $\{s\}$, and $\{t\}$ each corresponding to a 1-cell. Hence, we have three kinds of edges in our complex.
- $\{r, s\}$, $\{s, t\}$, and $\{r, t\}$ each corresponding to a 2-cell. Hence, we have three kinds of faces in our complex.

Then we look at the cosets for each T-cell, as in our previous example:

- W_\emptyset has cosets: $W_\emptyset, rW_\emptyset, sW_\emptyset, tW_\emptyset, (rs)W_\emptyset, (rt)W_\emptyset, (st)W_\emptyset, (rst)W_\emptyset$. Each coset is identified with a vertex.
- $W_{\{r\}}$ has cosets: $W_{\{r\}}, sW_{\{r\}}, tW_{\{r\}}$ and $(st)W_{\{r\}}$, each corresponding to edges with vertices $(\emptyset, r), (s, sr), (t, tr)$ and (st, str) .
- $W_{\{s\}}$ and $W_{\{t\}}$ have cosets similar to $W_{\{r\}}$, but for $\{s\}$ and $\{t\}$.
- $W_{\{r,s\}}$ has two cosets: $W_{\{r,s\}}$ and $tW_{\{r,s\}}$. Each corresponds to a face in the CW-complex.
- $W_{\{s,t\}}$ and $W_{\{r,t\}}$ have cosets similar to $W_{\{r,s\}}$.

Gluing vertices, edges and faces together we obtain a Σ_{cc} for G that is a CW-complex corresponding to a cube. Note that the complex is isomorphic to the two-sphere S^2 , which agrees with our conclusions from Example 4.6. The cellulation is shown in Figure 4.10. We also observe again that the 1-skeleton of Σ_{cc} has a 1:1 correspondence with the Cayley diagram for this group (see Figure 3.5).

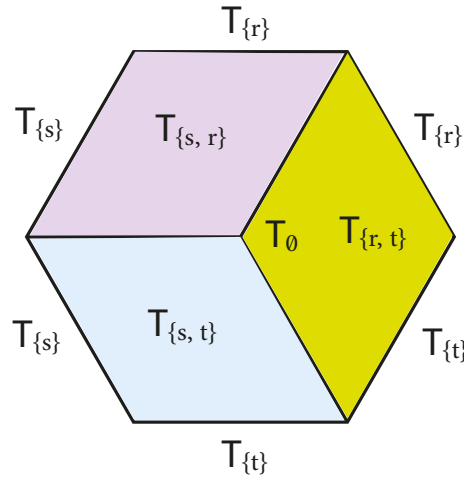


Figure 4.10: Partial illustration for the Coxeter cellulation Σ_{cc} for the group of Example 4.10. Being a cube a 3-d shape, we can only visualise three of its faces. The other three faces, and corresponding vertices and edges, are hidden in this figure.

Proposition 4.2.1. *The 1-skeleton of Σ_{cc} is isomorphic to $\text{Cayley}(W, S)$.*

Proof. The 1-skeleton of Σ_{cc} is built from the Coxeter cells of size 1, $|T| = 1$, to which correspond cosets aW_s , for s the generator of the T-cell and $a \in S$. This connects a to as and thus acts as an operation of a generator. By definition, this process builds a Cayley graph as outlined in Section 2.1.4. \square

Conclusions

This dissertation provides an introduction to Coxeter groups geometrically and topologically. Chapter 2 – **Background** – provides the reader with the necessary tools in group theory, geometry, and topology to tackle the main content of this work. Chapter 3 – **Coxeter Groups** – introduces Coxeter groups in terms of their group presentations and it discusses how they manifest in different settings through Coxeter-Dynkin diagrams and Cayley graphs. A brief discussion about the classification of finite groups follows based on the seminal work by Coxeter in the 1930s. Finally, Chapter 4 – **Coxeter Cellulations** – focuses on the cellulation of Coxeter groups via Davis complexes Σ . In particular, we investigate different ways to construct a Davis complex and show through examples how different groups are cellulated in the Euclidean, spherical and hyperbolic spaces.

The mathematics emerging from the theory of Coxeter groups is extensive and finds applications into other disciplines too. Geometric group theory and Coxeter groups have influenced the arts, as we have seen in some of M.C. Escher’s art, and also provide a framework to explain the geometric principles behind the art itself. Notably, colour symmetries on Coxeter groups can explain how one can colour particular tessellations while preserving other given group relations.

Beyond art, we also see applications of Coxeter groups in the sciences. In quantum and particle physics Coxeter groups are used in the study of Lie algebra, while in chemistry in the understanding of atomic structures via crystallography. Within the scope of mathematics, one could delve deeper into the geometry of Coxeter groups of non-Euclidean spaces, high-dimensional groups, or into their manifestations in root systems.[Hum90, Dav12] Another rich area of exploration is the study of Davis complexes Σ and in particular their role as CAT(0) spaces. These are spaces where triangles are “thinner” than in the Euclidean space.³¹ and Davis Complexes are non-positively curved, thus possessing geometric rigidity that is of interest to researchers in group theory, topology, as well as in computing and other sciences.[Dav94, Dav12, BH13].

³¹This means that any two points on opposite edges of a triangle are less than or equal in distance than they would be in an equivalent Euclidean triangle.

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Chapter 1

Figure 1.1 shows four woodcuts from M. C. Escher. These images are protected by *Fair Use Copyright*. Circle Limit I, Circle Limit III, and Circle Limit IV have been retrieved from <https://wikiart.org> – *Visual Art Encyclopedia*. Circle Limit II has been retrieved from <https://eschermath.org> – *Math and the Art of MC Escher*.

Chapter 2

Figure 2.8 of the Cayley graph of F_2 is from *Wikimedia Commons* and released into the *public domain*.

Figure 2.10 of a Poincaré disc is from *Wikimedia Commons* and released into the *public domain*.

Chapter 3

Figure 3.12 of the full enumeration of finite groups is from [Cox35].

Chapter 4

Figure 4.1 of the spherical tessellation of $(2, 2, 2)$ is from *Wikimedia Commons* and released under the *Creative Commons CC0 1.0 Universal Public Domain Dedication*.

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Appendix

Figure A.1 of an Hasse diagram is from *Wikimedia Commons* and released under the *GNU Free Documentation License*.

Appendices

Set Theory

Definition A.1 (Open set). Let S be a subset of a metric space. Then the set S is **open** if every point in S has a neighborhood lying in the set. More generally, given a topology (consisting of a set X and a collection of subsets τ), a set is said to be open if it is in τ . [Weib]

Definition A.2 (Power Set). The **power set** is the set of all the subsets of a set.

Definition A.3 (Partially Ordered Set). A **partially ordered set** or **poset** is a pair $\langle S, G \rangle$, where S is a set and G is an order relation for S . The order G is often represented using the symbol \leq and the following axioms must be satisfied:

- **(Reflexivity)**: $x \leq x$
- **(Antisymmetry)**: $x \leq y$ and $y \leq x \implies x = y$
- **(Transitivity)**: if $x \leq y$ and $y \leq z$, then $x \leq z$

A poset can be visualised via an **Hasse diagram** which consists of:

- A vertex for each element of the set S .
- An edge for each relation $x < y$.

An Hasse diagram is drawn with an upward orientation, such that “greater” elements are at the top.

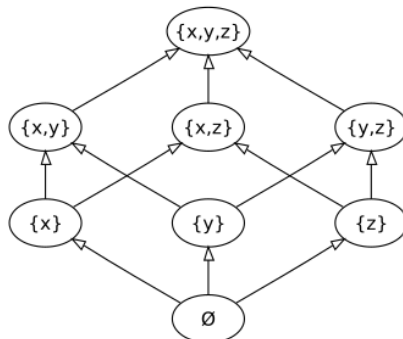


Figure A.1: Hasse diagram for the power set of $\{x, y, z\}$ with order \subseteq .

Group Theory

Definition B.1 (Direct Product). Given two groups (G, \circ) and (H, \star) , their **direct product** $G \times H$ is a new group, such that:

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

and with binary operations:

$$(g, h) \cdot (g', h') = (g \circ g', h \star h')$$

Definition B.2 (Free Product). Given two groups (G, \circ) and (H, \star) , their **direct product** $G * H$ is a new group, such that G and H do not necessarily need to commute and its presentation can be written as the union of their generators and relators:

$$G * H = \langle S_G \cup S_H \mid R_G \cup R_H \rangle$$

Definition B.3 (Conjugate Element). Given $g \in G$, its **conjugate** is $xgx^{-1} \in G$, such that $x \in G$.

Definition B.4 (Coset). Let G be a group and $H \leq G$. A **coset** is defined as:

$$gH = \{gh \mid h \in H\}$$

for some $g \in G$

Definition B.5 (Normal Subgroup). Let H be a subgroup of G . H is a **normal subgroup** if it is closed with respect to conjugates:

$$ghg^{-1} \in H; \quad \forall g \in G, h \in H$$

We denote it as: $H \triangleleft G$

Definition B.6 (Group presentation).

$$\langle S \mid R \rangle = F_S / \langle\langle R \rangle\rangle$$

- A group is finitely generated if S is finite.
- A group is finitely related if R is finite.
- A group is finitely presented if it is finitely generated and related.

Theorem B.1 (First Isomorphism Theorem [DF04]). *Given a homomorphism:*

$$\phi : G \longrightarrow H$$

Then $\ker(\phi) \triangleleft G$ and $G/\ker(\phi) \cong \phi(G)$.

Proposition B.1. *Every group G is isomorphic to a quotient group of the free group F by a normal subgroup.*

Proof. Consider a group G generated by S and an homomorphism:

$$\phi : F_S \longrightarrow G_S$$

Then $\ker(\phi) = \{f \in F_S \mid \phi(f) = e_{G_S}\}$.

By the first isomorphism theorem, $G_S = \phi(F_S) \cong F_S/\ker(\phi)$. However, note that $\ker(\phi)$ is the normal subgroup of F_S such that its generators correspond to the relators of G .

Thus:

$$G_S \cong F_S/\langle\langle R \rangle\rangle$$

□

Graph Theory

A **graph** is a pair of sets $G = (V, E)$, with V being the vertices and E the edges. The set V must be non-empty. An edge is a pair of distinct vertices from the set V . Two vertices connected by an edge, are said to be adjacent.

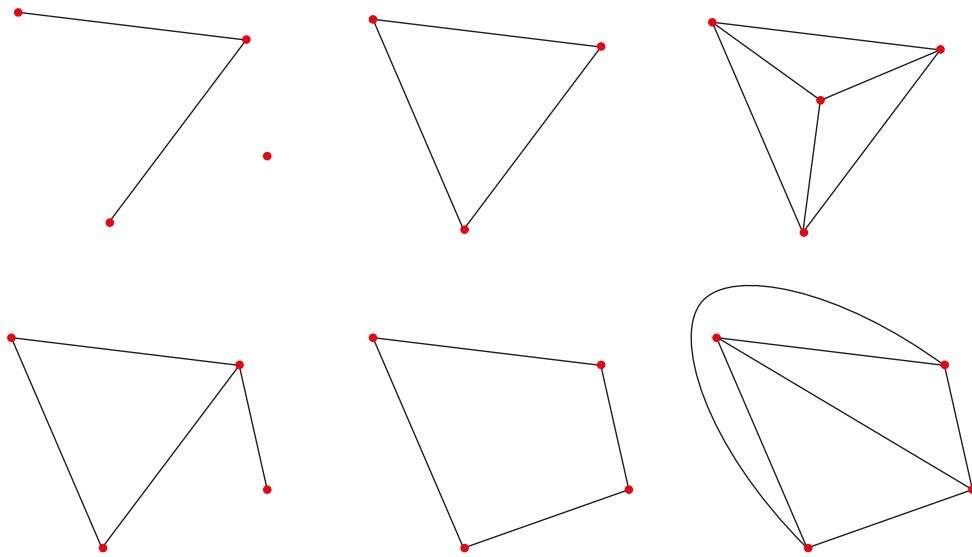


Figure C.1: Examples of graphs.

Two graphs G and H are **isomorphic** if there exists a bijection $f : V_G \rightarrow V_H$, such that if two vertices $v_{G_1}, v_{G_2} \in V$ are adjacent, then $v_{H_1}, v_{H_2} \in V_H$ are also adjacent. We write $G \cong H$.

Definition C.1 (Complete). A graph is **complete** if any two vertices are adjacent.

Definition C.2 (Bipartite). A graph is **bipartite** if its vertices can be divided into two subsets such that vertices within each set are not adjacent to other vertices within that set.

Definition C.3 (Complete Bipartite). A graph $K_{p,q}$ with two subsets of p - and q -vertices is **complete bipartite** if every vertex in a set is adjacent to every other vertex in the other set.

For more information about graph theory, [Tru93] is a good first introduction to it.

Frequently Used Notation

\mathbb{Z}	the integers $(\dots, -3, -2, -1, 0, 1, 2, 3, \dots)$
\mathbb{R}	the reals
\mathbb{E}^n	the n -dimensional Euclidean space
\mathbb{S}^n	the n -dimensional spherical space
\mathbb{H}^n	the n -dimensional hyperbolic space
\exists	there exists (the existential qualifier)
\forall	for all (the universal qualifier)
$!$	unique
$A \subseteq B$	A is a subset of B
\circ or \star	binary operation of a group
\sim	binary relation
\cdot	group action
$\langle S R \rangle$	group generated by the set S and with relators in R
$G \times H$	Direct product between group G and group H
$G * H$	Free product between group G and group H
e	identity element in a group
(G, \circ)	group G with operation \circ
$H \leq G$	H is a subgroup of G
$H \triangleleft G$	H is a normal subgroup of G
H_U	subgroup H generated by elements of U
$A \cong B$	A is isomorphic to B
Σ	Davis complex
Σ_{cc}	Coxeter cellulation for a Davis complex Σ

Common Groups

$O(n)$	the group of orthogonal transformations in \mathbb{E}^n that preserve points in space
$GL_n(\mathbb{F})$	the group of $n \times n$ invertible matrices with elements from a field \mathbb{F}
S_n	symmetric group with n -elements
D_n	dihedral group with n -sides and $2n$ symmetries
\mathbb{Z}_n	the group of integers modulo n
C_n	the cyclic group with n -elements, isomorphic to \mathbb{Z}_n