What are tensors and how can they be applied to mechanics?

Subject area: Mathematics

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Word Count: 3968

Abstract

The aim of my extended essay is to give an insight of the tensors, some of their features and how they can be applied to one of the most important branch of physics of the last fifty centuries: mechanics. The essay begins with a general remark on the importance of mathematics on everyday life and how I have been introduced to tensors in my early life. After having looked up to how the first mathematicians arrived to develop tensors' theory, I defined tensors comparing them to vectors and explaining some of their most important characteristics. Then I connected the mathematical part of the tensors to mechanics and to the concept of stress. This section includes the orthonormal transformation which connects to the second part of the essay about the Mohr's circle. The Mohr's circle is important for analyzing the stress of bodies that are subjected to rotation. First of all the topic is treated on the two-dimensional space, then a third dimension is introduced to make the Mohr's circle extremely useful in real life. At the end of my investigation I found out why the basic understanding of tensors and the Mohr's circle are considered such good tools by physicists and engineers.

I used the software Kig v. 1.0 to prove and make more understandable all the concepts and the procedure that I did. All the assumptions that appear in this essay are present in my sources and helped me to understand and develop the methods used. In the case of the Mohr's circle I also plug in numbers to show its real application by physicists.

Word count: 266

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Introduction

Most of the people who do not study scientific subjects or not pursue a job that implies calculations have a really interesting view about Mathematics. At the pronunciation of any word that begins with MATH- their first thought is about an obscure and abstractive set of rules, formulas, and numbers that pretend to be the most important among the all sciences. However, not having a good understanding of what mathematics is they cannot have a concrete concept of its usefulness. Mathematics, in fact, is “the science that draws necessary conclusions”[1] and that it can always be used to find any kind of solution not only on a paper test, but in the real world as well. Moreover, mathematics is crucial in many other subjects, such as: physics, biology, chemistry. Tensors' theory is one of the area of mathematics that easily find applications in everyday life, expecially in the branch of mechanics.

The research question of my essay is:

What are tensors and how can they be applied to mechanics?

When there is a strong wind or earthquake buildings are subjected to strong solicitations. Solicitations are forces exerted on any kind of object and they are usually known as stress tensors. I will investigate some of the main properties of tensors and why they are so important in mechanics. First of all I will consider a pure mathematical approach which implies the definition of tensor, its matrix representation and transformation. Then I will use a more geometrical approach, the Mohr's circle, mostly used by engineers to demonstrate their usefulness to understand many physical phenomena.

1 Benjamin Peirce, American Journal of Mathematics, Vol. 4, No. ¼, “Linear Associative Algebra”, John Hopkins University Press, United States, 1881

From the creation to today's tensors' theory

In the nineteenth century the French mathematician Augustin-Louis Cauchy contributed enormously in mathematics and mathematical physics, which is the study that relates mathematics with physics. Its definition from the *Journal of Mathematical Physics* states that it is "the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories."[2]. Since the beginning of his career Cauchy focused principally on physics and how it could be explained mathematically. Thus, he started developing the tensors' theory, applying it to real world problems. In the early 1822 Cauchy introduced the term stress tensor to the scientific community. In classic mechanics Stress is a dynamic quantity that expresses the magnitude of forces acting within or at the surface (*Illustration 1*).

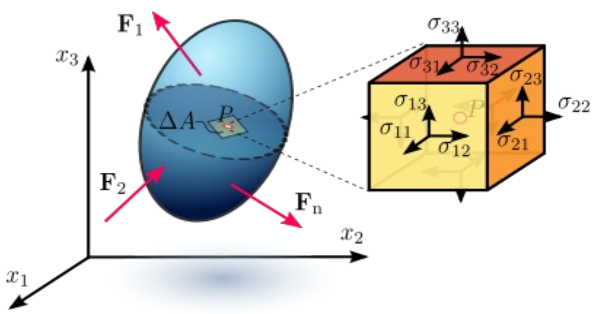

Illustration 1:

But the science of tensors existed before Cauchy introduced his work in the mathematician's community. Tensors were firstly created by the two German mathematicians Bernard Riemann and Elvin Bruno Christoffel in the year 1827. Later on the two other scientists Tullio Levi-Civita and Gregorio Ricci-Curbastro went further in the study of tensors formulating the differential geometry of a

2 Journal of Mathematical Physics, About the Journal, Focus and coverage, Website, accessed 4.57 pm, 11 November 2009, <http://jmp.aip.org/jmp/staff.jsp>

manifold in the form of the Riemann curvature tensor.

According to Cauchy's work the stress applied at any point of an object is defined by a tensor of second order which name is Cauchy's stress tensor[3]. However he was able to use tensors just for the stress analysis of objects experiencing small deformations. Successively the physicists and mathematicians Gustav Robert Kiachhoff and Gabrio Piola extended Cauchy's stress tensor to large deformations of materials.

Nowadays the work of these 19th century's scientists on tensors and their application in stress analysis is used intensively in engineering for the study and design of structured, bridges, tunnels, and mechanical parts among others.

A multidimensional world

In the trilogy '*The Illiminatus*!' the sentence "*There are many multiverses, each with its own dimensions, times, spaces, laws and eccentricities*"[4] tells how physicists see the set of elements all around them. If they are able to perceive such a state of things is because mathematics has given them the possibility to do so.

For example when someone stand in a place he or she can be mathematically expressed as a point. If the person move forward in one direction he is defining a line, and so on. Step by step, the person is jumping from the zeroth dimension into a multidimensional space.

The first dimension is a number which expresses a magnitude. By using more points it is derived the concepts of the second dimension. By connecting two points in a plane it is got the line, which has scalar properties, and observing the line segment with direction and versus I get to the vector. A scalar

3 Wikipedia The free Encyclopedia, 'Stress mechanics', Website, accessed 3.06 pm, 19 September 2009 , <http://en.wikipedia.org/wiki/Stress_(mechanics)>

4 Robert Shea and Robert Anton Wilson, 'The Illuminatus! Trilogy', Dell Publishing, New York City, 1975, p. 47

is what is known as Tensor of rank 0 in tensors' theory. Vectors, or tensors of rank 1, have three dimensions on the space and have magnitude, direction and versus (*Illustration 2*). A vector can give many informations, that is why is the most used mathematical tool by physicians.

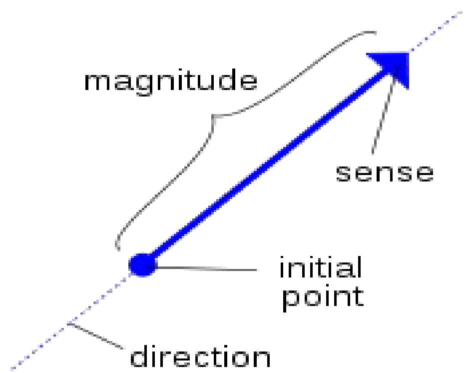

Illustration 2

A vector ***v*** is geometric object that is based on the combination of a scalar (magnitude) and a vectorial (direction and sense) part and it is usually expressed as:

$\boldsymbol{v} = a_i + b_j + c_k$, where **a**, **b**, and **c** are the scalar part of the vector, while **i**, **j** and **k** are its unit vector and consequently indicate the direction where the vector is pointing to in the three dimensional space.

Going further we encounter what is called: tensor. *A tensor is defined as a geometrical entity that expresses the relationship between the input and output vectors and provide a natural and concise mathematical framework for formulating and solving problems in areas of physics such as elasticity, fluid mechanics, and general relativity*[5]. However tensors do not define a fourth dimension. In fact they are always built on three dimensions, but they are the combinations of two or more vector and posses interesting physical properties.

A tensor of rank 2 is called Dyad and, differently from a vector or a scalar, has a magnitude and two

5 B.M. Budak and S. V. Fomina, 'Multiple Integrals, Field Theory and Series – An Advanced Course in Higher Mathematics', MIR Publishers, Moscow, 1973, p. 274

directions. Thus, a dyad is formed by three sub-vectors. Each sub-vector is formed by three components, thus a dyad is generally formed by nine components on a Cartesian plane. A dyad ***UV*** is the combination of the two vectors ***v*** and ***u***. By the way, with the expression ***UV*** is not indicated even a product or a cross product of the two vectors, but just a distinct entity. So, if ***u*** **=** $u_1 i+u_2 j+u_3 k$ and ***v*** **=** $v_1 i+v_2 j+v_3 k$,

therefore ***UV*** = $u_1 v_1 ii+u_1 v_2 ij+u_1 v_3 ik+\ldots+u_3 v_3 kk$

As shown above ***UV*** is a really long expression, but I can simplify it by writing $u_m v_n mn$ as σ_{mn} .

In the case of the vectors ***u*** and ***v*** the unit vectors **i**, **j**, and **k**, which are special vectors with a length of one unit, indicate the direction of the component respect to the X, Y, and Z axes respectively. Similarly unit dyads are used for tensors. As it has been previously mentioned a tensor has itself three sub-vectors. The first 'unit vector' of the unit dyad indicate the sub-vector. So **i** indicates the first one, **j** the second one and **k** the third one. The second 'unit vector' of the unit dyad indicate the axes on which the single component lies as in the case of the vectors ***u*** and ***v***. The numbers *m* and *n* are used in parallel to the unit dyad and they can be: 1, 2, and 3.

so the dyad is expressed as: ***UV*** = $\sigma_{11} ii+\sigma_{12} ij+\sigma_{13} ik+\ldots+\sigma_{33} kk$.

One feature of this dyad is that the nine components of ***UV*** can be represent as a 3x3 matrix and its graphical representation is given by the illustration 3:

$$UV=\begin{matrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{matrix}$$

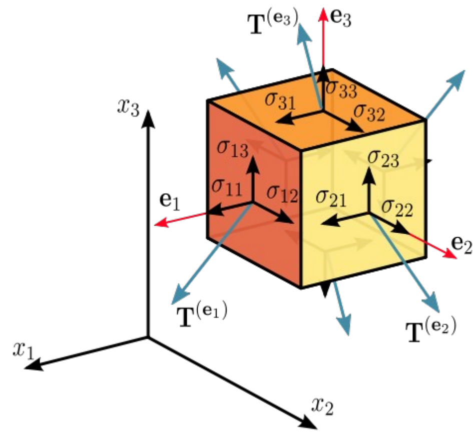

Illustration 3

In the illustration above $T^{(e_m)}$ is the stress vector , where m = 1,2,3 is the index of the orthonormal basis **e** and indicates the plane of the relative stress vector. The general stress vector $T^{(e_m)}$ is equal to the sum of the stress vectors of each plane.

$$T^{(n)}=T^{(e_1)}n_1+T^{(e_2)}n_2+T^{(e_3)}n_3$$

where T^{e_1} is the first row, T^{e_2} is the second row , and T^{e_3} is the third row of the 3x3 matrix ***UV***.

The orthonormal transformation

An orthonormal base **e** is defined as a subset $\{v_1,v_2,v_3\}$ of a three dimensional vector field ***v*** and in its simplest form $\mathbf{e}_i$ is a standard base into a Euclidean space made by a set of real numbers. Since vector are perpendicular each other I obtained the following relationships between the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$:

$$\mathbf{e}_1 \bullet \mathbf{e}_1 = 1 \; ; \quad \mathbf{e}_1 \bullet \mathbf{e}_2 = 0 \; ; \quad \mathbf{e}_1 \bullet \mathbf{e}_3 = 0 \; ;$$
$$\mathbf{e}_2 \bullet \mathbf{e}_1 = 0 \; ; \quad \mathbf{e}_2 \bullet \mathbf{e}_2 = 1 \; ; \quad \mathbf{e}_2 \bullet \mathbf{e}_3 = 0 \; ;$$
$$\mathbf{e}_3 \bullet \mathbf{e}_1 = 0 \; ; \quad \mathbf{e}_3 \bullet \mathbf{e}_2 = 0 \; ; \quad \mathbf{e}_3 \bullet \mathbf{e}_3 = 1 \; .$$

The dot, •, indicates a scalar product between two orthonormal basis $\mathbf{e}_i$. Thus the product gives us 1 when the vector is multiplied by itself, otherwise it gives 0 because the base vectors are perpendicular to each other.

Following the rules that govern the most simple vectors in a two or three dimensional space, if I rotate $\mathbf{e}_i$ its orthonormal set changes. These kind of transformations are known as orthogonal transformations. The base **e** can then be considered as a vector and split in its simpler components: x and y. Because of the rotation the x component of **e'** will depend on the y component of **e**, whose x coordinate will not be 0 anymore. Similarly the y component of **e'** will depend on the x component of **e**, which previously had coordinates (1,0). Thus the orthonormal transformation is written as follow.

(*Matrix 1*)

$$e'_1 = \alpha_{11}e_1 + \alpha_{12}e_2$$
$$e'_2 = \alpha_{21}e_1 + \alpha_{22}e_2$$

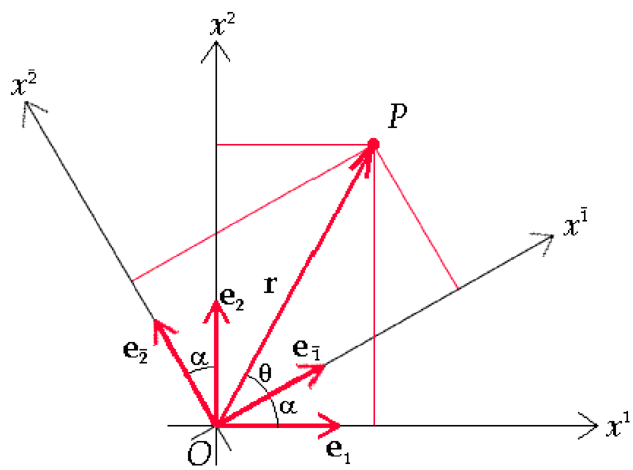

Figure 2

Illustration 4:

As it is shown in the Illustration 4 the components $\mathbf{e_1}$ and $\mathbf{e_2}$ (or the x and y components referring to the Cartesian plane) are rotated counterclockwise about the origin by an angle α.

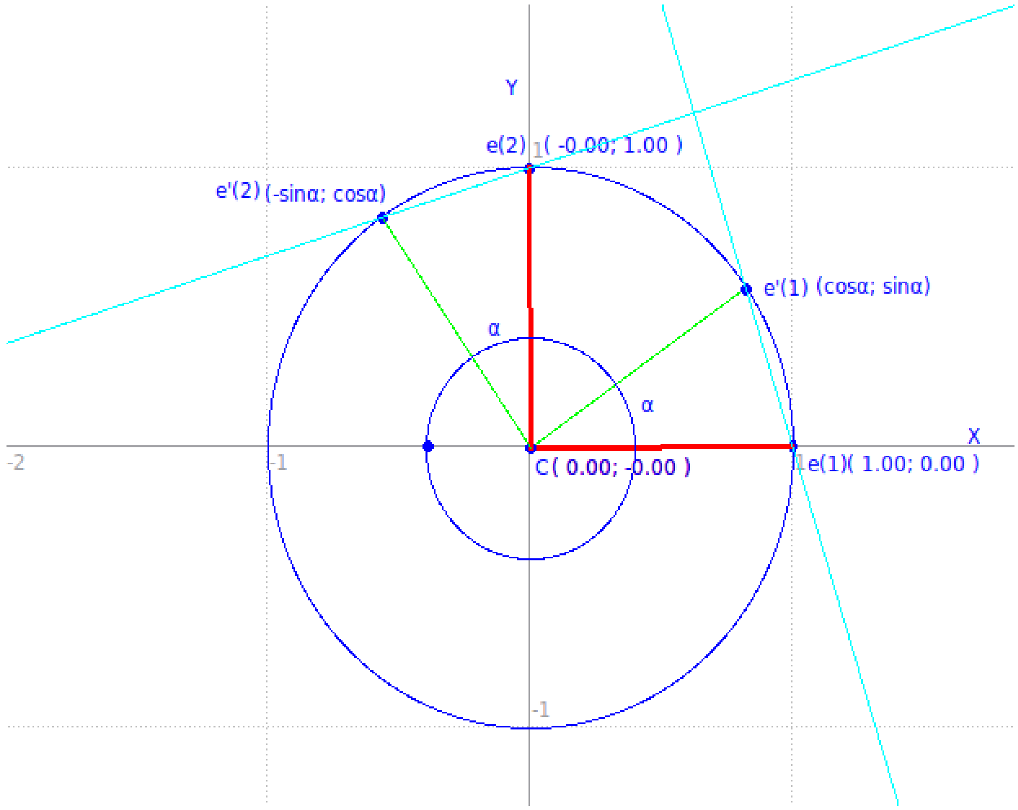

Illustration 5:

The illustration 5 shows the two basis on the x-y Cartesian plane where **e**$_1$ (1,0) and **e**$_2$ (0,1). These are two points that lie on the circumference of a unit circle. Now I find the coordinates of these two points after they rotate and we multiply them by the x and y coordinates of the base **e**.

When the point **e**$_1$ rotates counterclockwise by an angle α, its new coordinates will by x = cosα and y = sinα. While the point **e**$_2$ will have coordinates x = -sinα and y = cosα.

Therefore the matrix for the points **e**$_1$ and **e**$_2$, which are part of a unit circle, is:

$$\begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix}.$$

However, because the base has its own x-y coordinates, I must multiply such coordinates for the matrix:

$$\begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_1' \\ e_2' \end{bmatrix}$$

An orthogonal transformation of ***v*** is written as ***u***: ***v*** → ***v'*** [6]. The matrix itself contains some physical properties of the object and when an external force acts on it the properties of the object change and consequently the matrix changes as well. It observed a transformation of the matrix from the standard orthonormal basis e_1, e_2, e_3 , of a three-dimensional Euclidean space, to the new ones: e'_1, e'_2, e'_3 [7].

The orthonormal transformation in terms of its orthonormal basis **e** is expressed linearly by using the Einstein convention as:

$$\mathbf{e}'_i = \Sigma\, \alpha ij\mathbf{e}j \text{ , where } i = 1, 2, 3$$

ll me the transformation of the tensor occurred. Here below is the representation of this matrix:

If I expand it I will obtain a 3x3 matrix that tells me the transformation of the tensor occurred. Here below is the representation of this matrix:

(*Matrix 2*)

$$e'_1 = \alpha_{11}e_1 + \alpha_{12}e_2 + \alpha_{13}e_3$$

$$e'_2 = \alpha_{21}e_1 + \alpha_{22}e_2 + \alpha_{23}e_3$$

$$e'_3 = \alpha_{31}e_1 + \alpha_{32}e_2 + \alpha_{33}e_3$$

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6 WolframMathworld. '*Orthogonal transformation', Website, accessed 10.48 pm, 25 September 2009,* <http://mathworld.wolfram.com/OrthogonalTransformation.html>

7 B.M. Budak and S. V. Fomina, 'Multiple Integrals, Field Theory and Series – An Advanced Course in Higher Mathematics', MIR Publishers, Moscow, 1973, p. 284

which is basically equal to $\boldsymbol{u}$: $\boldsymbol{v} \rightarrow \boldsymbol{v}'$. Omitting the orthonormal basis such matrix transformation can be expressed in terms of α (alpha).

$$\alpha = \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{matrix}$$

Briefly the orthonormal transformation of a 3x3 matrix is graphically similar to the transformation of a 2x2 matrix. The only difference is that a third dimension is not included. In the illustration 3 is shown an orthogonal transformation on a 2-dimensional Cartesian plane, where all the points are rotated by the same angle α.

Mohr's circle

In stress theory the transformations of tensors are continuously applied to the problems analyzed. In 1882 the German civil engineer Christian Otto Mohr developed a graphical method for analyzing a coordinate transformations of stress at a certain point. This method is well known as Mohr's circle and because of his simplicity is extremely used everyday. This technique can be applied to both symmetrical 2x2 and 3x3 tensor matrices. In brief Mohr’s circle is extremely valuable as a quick graphical estimation tool to double-check an analytical work. A 2x2 tensors matrix is the most used case of a Mohr's circle.

Lets suppose we have a matrix $\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

these four components refer to some specific basis that is possible to represent geometrically (*Illustration 6*):

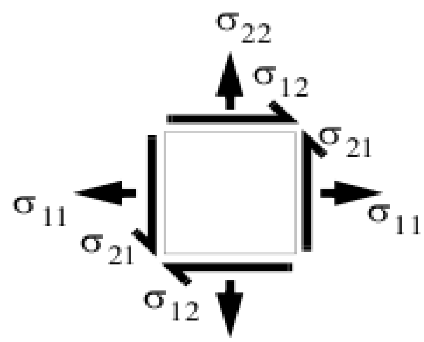

Illustration 6:

Mathematicians usually express it as the matrix 2x2:

$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a = σ_{11}, b = σ_{12}, c = σ_{21}, d = σ_{22}

So the Illustration 4 can be sketched in a simplified way as shown below (*Illustration 7*):

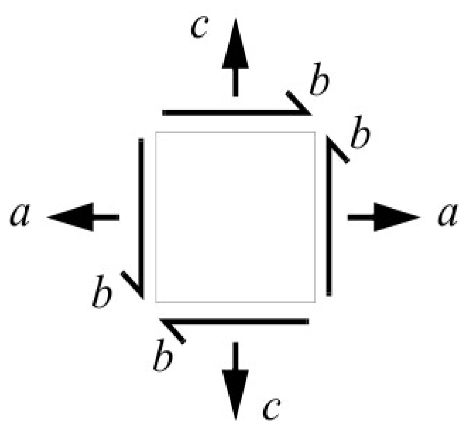

Illustration 7:

This square can be sectioned in two sides: H side (The face whose outward unit normal is horizontal) and V side (the face whose outward normal is vertical). Therefore, from the figure above (*Illustration*

7), H-face is equal to **a**, and V-face is equal to **c**. Where a, b, and c are components that define the normal stress of a body. These components are considered positive if they are in tension, and negative if they are in compression.

Always taking a look to the Illustration 7 it is possible to notice how the H and the V sides have a component in common. This is called the shearing component or shearing stress and it is given a numerical sign to it in accordance to the left-hand rule. Thus, the shearing stress on the H-face equals –b while the shearing stress on the V-face equals +b . The negative sign of the shearing stress on the H-face is explained by the third law of Newton which states that to every action there is an equal and opposite reaction (*Illustration 8*).

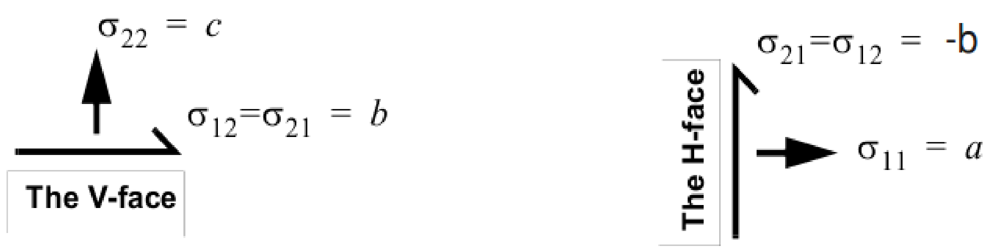

Illustration 8:

The matrix **A** taken in consideration is symmetric so the shearing stresses on H and V faces balance each other so that there is no net torque. Torque is the tendency of a force to rotate an object about an axis.

When a body is under stress it rotates. Always considering the square, I rotate it by an angle θ (delta) respect to the x-axis of a 2-dimensional plane (*Illustration 9*). I am facing a coordinate transformation. Drawing the rotation of the body the figure below is obtained. From such transformation, physicists aim to know what the normal and shear stress will be after the rotation of the body occurred.

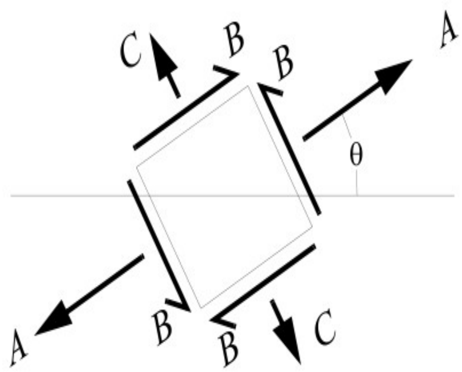

Illustration 9:

Conventionally the normal is indicated by the symbol σ (sigma), while the other Greek letter τ (tau) is used for the shear. So on the H side the normal stress σ is equal $\sigma_{11} = a$, while the share stress τ is equal to $-\sigma_{21} = -b$. From the other hand on the V side the normal stress σ is equal $\sigma_{22} = c$, while the share stress τ is equal to $\sigma_{12} = b$.

Therefore the coordinate point for the H and V sides are:

H: (a, -b)

V: (c, b)

Normally to define a circle three point are needed, but Mohr says that *"Whenever the orientations of two planes differ by exactly 90 degrees, the corresponding points on Mohr's circle will be diametrically opposite each other. "*[8]

Now defining a Cartesian plan whose abscissa indicates the values for the normal stress σ and whose ordinate is used for indicating the shear stress τ, I can represent the initial position of the body graphically (*Illustration 10*):

8 Rebecca Brannon, Essay 'Mohr's circle and more circles', Mohr's circle for symmetric matrices, p. 9

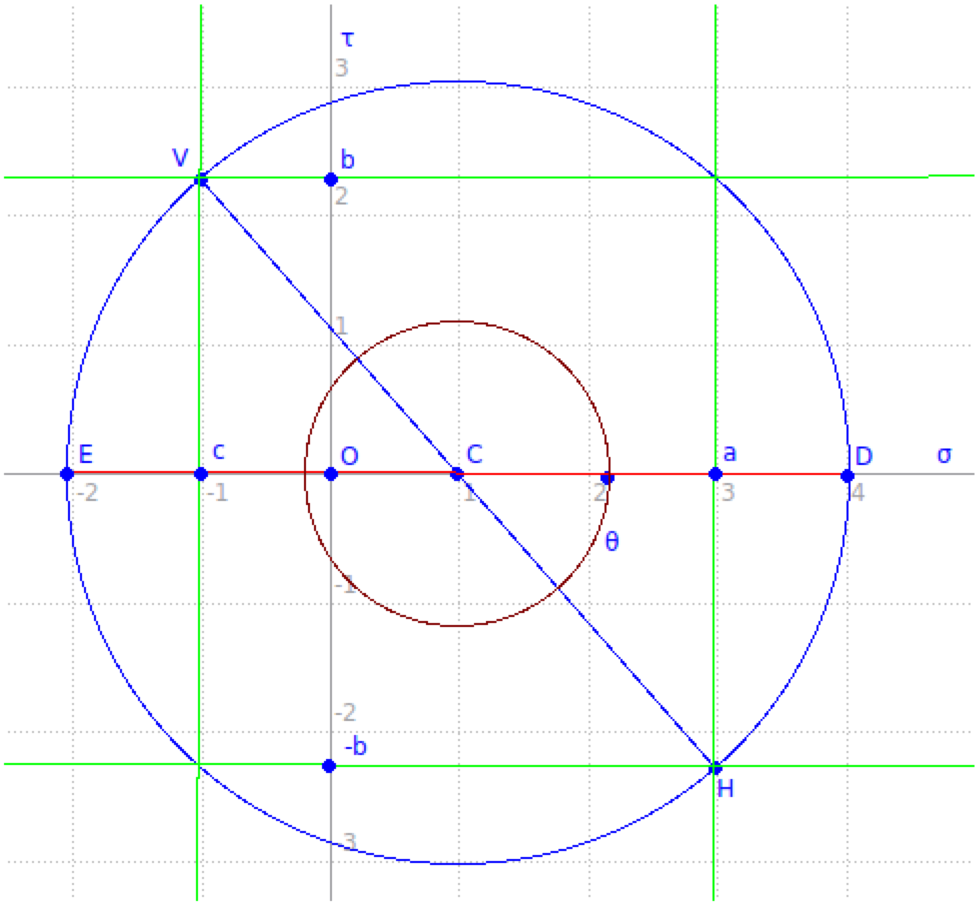

Illustration 10:

The maximum stress is obtained by rotating H around the center C for an angle θ (between CH and σ-axis) so that the point OD is the maximum stress for the point H. For the point V applies the same rule, but its maximum stress is at EO (*Illustration 10*).

Finding stress for any angle θ demands the following steps, that are shown within an example. The stress matrix that I considered is:

$$\mathbf{M} = \begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix}$$ and it rotates by an angle θ of 20°.

From the theory above it follows that the H and V coordinates of the matrix are:

H: (4, -4)

V: (-2, 4)

The point H and V are visible in the Illustration 11.

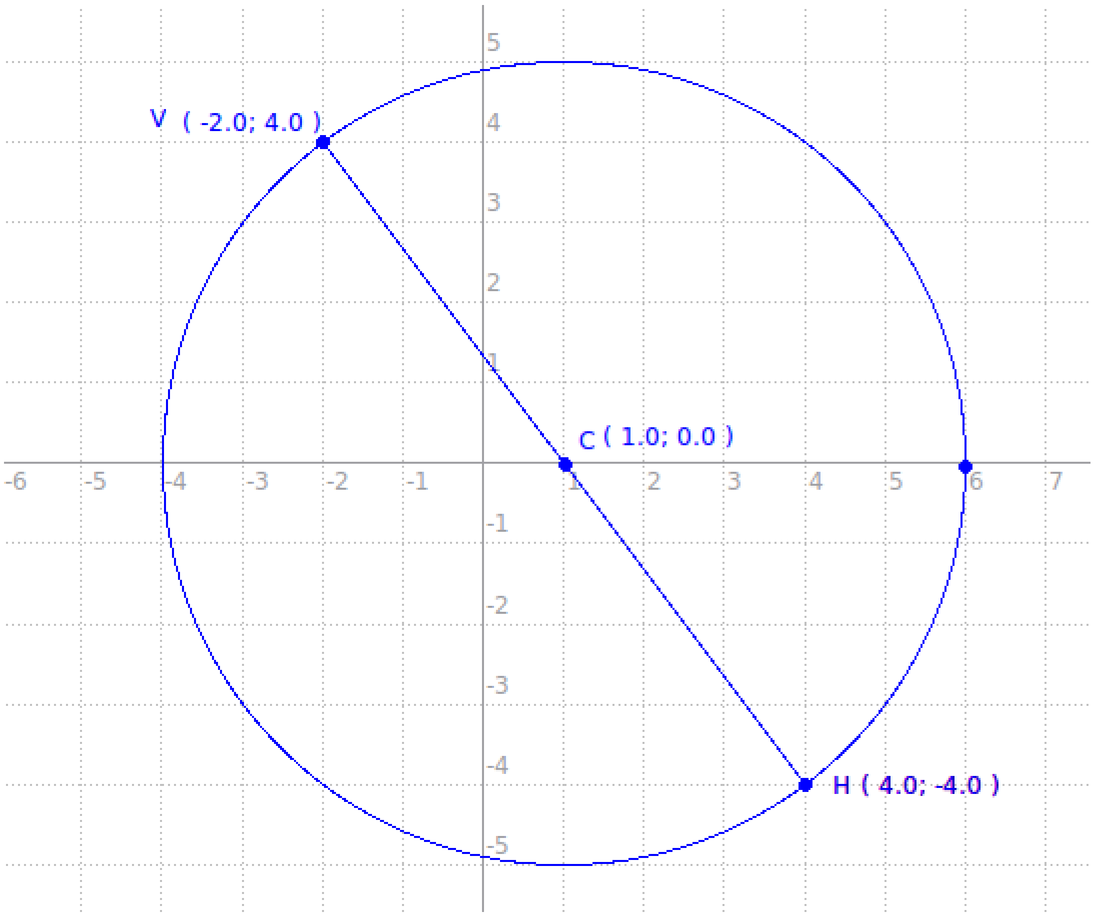

Illustration 11:

Now I measure the angle of rotation (20°) counterclockwise from the point H to obtain the point that has under gone the transformation H'. The same principle is applied to the point V, so that at the end I have a transformation H, V → H', V' , which is graphically explained in the Illustration 12. The new coordinates of these point can be calculated mathematically.

The new points H' and V' are find by using the following procedure:

From the location of the points H and V I extrapolate the coordinates for the center of the circle C.

C: (1, 0) (*Illustration 11*).

The standard formula that describes a circle with center C (a, b) in a Cartesian plane is

$$(x-a)^2+(y-b)^2=r^2$$

In my example (x - a) is equal to c, while (y – b) is equal to (H_Y-C_Y). Being C_Y on the x-axis, (y - b) becomes just H_Y.

Thus the Mohr's circle has formula:

$$r^2=(H_x-C_x)^2+(H_y-C_y)^2 \text{ or } r^2=(H_x-C_x)^2+(H_y)^2$$

So,

$$r = \sqrt{(H_X-C_X)^2+(H_Y-C_Y)^2} = \sqrt{3^2+4^2} = 5$$

α is the angle between the point H and the abscissa, the σ-axis.

From the figure above and some basic notions of trigonometry it is possible to find the value of α, knowing the σ and τ coordinates of the point taken in consideration respect to C. Projecting the point H on the σ-axis, H_1 we can find the angle α, from the triangle H_1CH.

$$\alpha=\arctan\left(\frac{4}{3}\right)\approx 53.13°$$

After the transformation occurs, the angle α decrease or increase by 2θ (40°). In the case I analyzed the object is rotated counterclockwise, so the new angle β is equal to α – (2 x 20°) = 13.13°.

After the transformation the point H and V are located into new coordinates. While I have a new angle β, the coordinates of the center C and the radius *r* of the circle are the same. Not being the center at coordinates (0,0) I have to add its x coordinate to the projection of the radius along the σ-axis. Meanwhile the y-coordinate for the new H point is just the projection of H_y along the τ-axis with a negative sign because the y coordinate of the Mohr's circle is 0.

Thus the new point H' has coordinates (C_X + rcos β, -rsinβ), while the point V' being symmetrically opposite to H' has coordinates (C_X - rcos β, rsinβ). By simple mathematical calculations:

H': (5.87, - 1.14)

V': (-3.87, 1.14)

H' and V' are graphically represented in (*Illustration 12*):

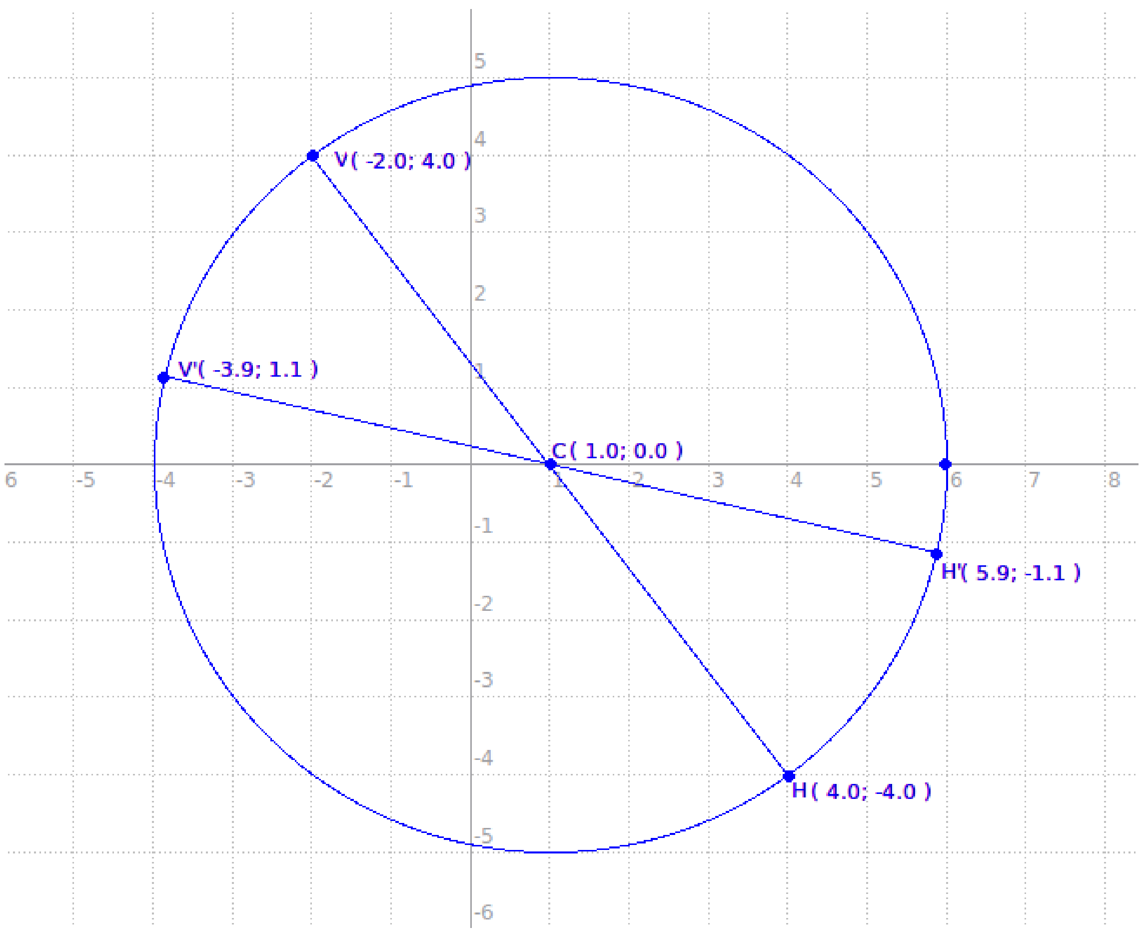

Illustration 12:

The new matrix describing the transformation is:

$$\begin{bmatrix} 5.87 & 1.14 \\ -1.14 & -3.87 \end{bmatrix}$$

To conclude if a force $\vec{F}$ is applied on an two dimensional object such that the point where the force is applied rotates of an angle α, the resultant stress will be described by a 2x2 matrix which includes the new values for the normal and share stress. This explains why tensors and the Mohr's circle are so important in engineering and physics.

Three-dimensional transformations by using Mohr's circle

Nevertheless the transformation of a stress tensor σ usually occurs on a three-dimensional space. Mohr's circle allow us to solve the transformation of a 3x3 matrix.

Lets consider the dyad ***UV*** antecedently taken in consideration:

$$\sigma = \begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix}$$

The first diagonal formed by the components: σ_{11}, σ_{22}, and σ_{33} describes the normal stress and I have assumed that their relationship is such that $\sigma_{11} > \sigma_{22} > \sigma_{33}$, so that these three values are placed on the Coordinate system as shown in Illustration 13. All the other components of the tensor σ describe the shear stress τ.

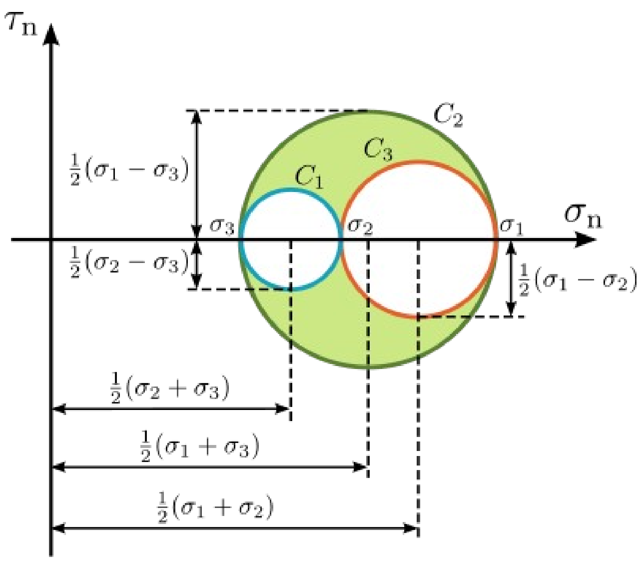

Illustration 13:

Considering the general term σ_{ij} , i and j are indicating its direction in the space. To make calculations simpler they can be expressed as **n**. So the normal stress α is equal to **σn.**

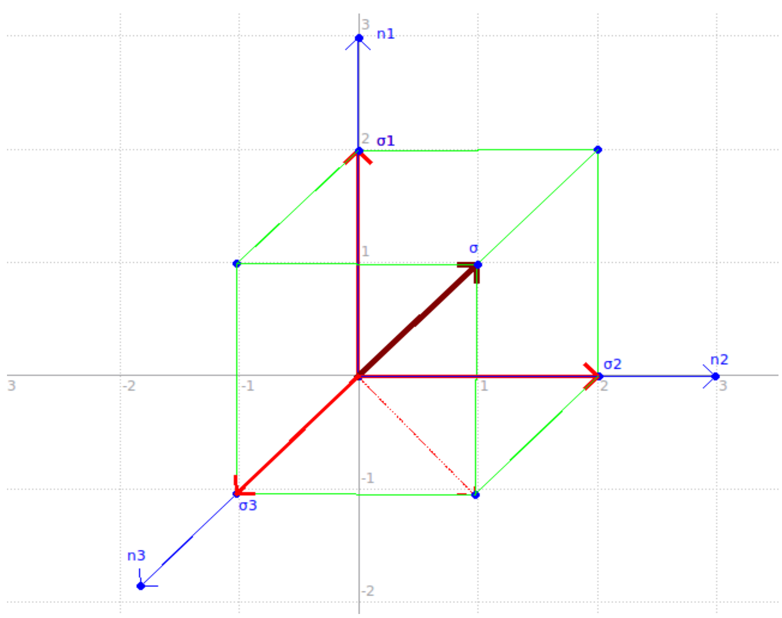

Illustration 14:

As I represented in illustration 14 the stress has three components, so $\sigma_1 \mathbf{n}_1^2 + \sigma_2 \mathbf{n}_2^2 + \sigma_3 \mathbf{n}_3^2 = \boldsymbol{\sigma}\mathbf{n}$

(*equation 1*) or by using the Einstein summation convention $\sigma n = \sum_{i=1}^{3} \sigma_i n_i^2$.

Where ***n*** is a vector which has a unit magnitude. Consequently its unit vector components must satisfy

$\boldsymbol{n}_1^2 + \boldsymbol{n}_2^2 + \boldsymbol{n}_3^2 = 1$ (*equation 2*)

their sum is equal one because when all the plane are perpendicular each other. In fact, when one coordinate of unit vector is 1, for a determined plane, all the others are equal to 0.

The shear stress, indeed, can be calculated by applying the Pythagoras theorem to stress vector $\mathbf{T}^{(n)}$ and the normal vector **n.** By conventionality $(\mathbf{T}^{(n)})^2 = \sigma_{ij}\sigma_{ik}n_j n_k$.

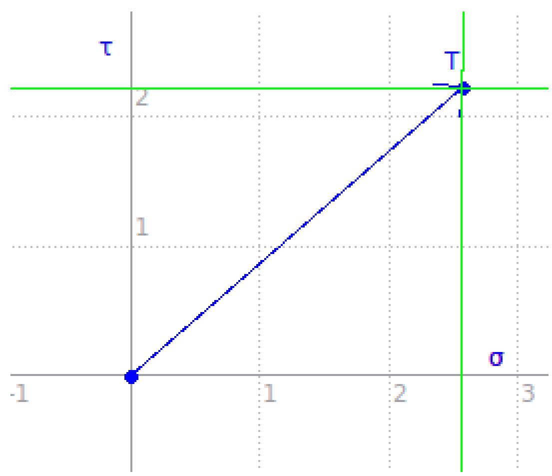

Illustration 15:

From the illustration 15 it is possible to see how $T^{(n)}$ is the hypotenuse of the triangle rectangle with catheti τ and σ. It follows:

$\tau = \sqrt{(\sigma \cdot n)^2 - \sigma^2}$ which, recalling the Einstein summation convention is equal to the expression below:

$\tau^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - \sigma^2$ (*equation 3*)

From the equations 1, 2, and 3 we can find out all the three ***n***s.

Solving it for $\boldsymbol{n}_1^2$ we obtain the follow expression:

$$\frac{\tau^2+(\sigma-\sigma_2)(\sigma-\sigma_3)}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)}\geq 0$$

$\boldsymbol{n}_2^2$, indeed, is equal to :

$$\frac{\tau^2+(\sigma-\sigma_3)(\sigma-\sigma_1)}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)}\leq 0$$

And at the end $\boldsymbol{n}_3^2$ is described by the equation:

$$\frac{\tau^2+(\sigma-\sigma_1)(\sigma-\sigma_2)}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)}\geq 0$$

According to the previous assumption that the correlation between the normal stress is $\sigma_{11} > \sigma_{22} > \sigma_{33}$; where in my example $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, and $\sigma_{33} = \sigma_3$; I have simplified the above three equations for $\boldsymbol{n}_1^2$, $\boldsymbol{n}_2^2$, and $\boldsymbol{n}_3^2$ as follows:

$$\tau^2+(\sigma-\sigma_2)(\sigma-\sigma_3)\geq 0$$

$$\tau^2+(\sigma-\sigma_3)(\sigma-\sigma_1)\leq 0$$

$$\tau^2+(\sigma-\sigma_1)(\sigma-\sigma_2)\geq 0$$

Lets expand the three inequalities and make suitable form, so that I will be able to compess them by usign the binomial theorem:

First inequality:

$$\tau^2+\sigma^2-\sigma\sigma_2-\sigma\sigma_3-\sigma_2\sigma_3\geq 0$$

$$\tau^2+\sigma^2+\frac{1}{4}\sigma_2^2-\frac{1}{4}\sigma_2^2+\frac{1}{4}\sigma_3^2-\frac{1}{4}\sigma_3^2-\sigma\sigma_2-\sigma\sigma_3+\frac{1}{2}\sigma_2\sigma_3+\frac{1}{2}\sigma_2\sigma_3\geq 0$$

as you can see inside in this inequality there are three expanded binomial expressions of second degree. So we compress these two expressions in their simple form to obtain:

$$\tau^2+\left(\sigma^2+\frac{1}{4}\sigma_2^2+\frac{1}{4}\sigma_3^2-\sigma\sigma_2-\sigma\sigma_3+\frac{1}{2}\sigma_2\sigma_3\right)-\left(\frac{1}{4}\sigma_2^2+\frac{1}{4}\sigma_3^2-\frac{1}{2}\sigma_2\sigma_3\right)\geq 0$$

$$\tau^2+\left[\sigma^2+\left(\frac{1}{4}\sigma_2^2+\frac{1}{4}\sigma_3^2+\frac{1}{2}\sigma_2\sigma_3\right)-\sigma\sigma_2-\sigma\sigma_3\right]-\left(\frac{1}{4}\sigma_2^2+\frac{1}{4}\sigma_3^2-\frac{1}{2}\sigma_2\sigma_3\right)\geq 0$$

$$\tau^2+\left[\sigma-\frac{1}{2}\left(\sigma_2+\sigma_3\right)\right]^2-\left[\frac{1}{2}\left(\sigma_2-\sigma_3\right)\right]^2\geq 0$$

Second inequality:

$$\tau^2+\sigma^2-\sigma\sigma_3-\sigma\sigma_1-\sigma_1\sigma_3\leq 0$$

$$\tau^2+\sigma^2+\frac{1}{4}\sigma_3^2-\frac{1}{4}\sigma_3^2+\frac{1}{4}\sigma_1^2-\frac{1}{4}\sigma_1^2-\sigma\sigma_3-\sigma\sigma_1+\frac{1}{2}\sigma_3\sigma_1+\frac{1}{2}\sigma_3\sigma_1\leq 0$$

$$\tau^2+\left(\sigma^2+\frac{1}{4}\sigma_3^2+\frac{1}{4}\sigma_1^2-\sigma\sigma_3-\sigma\sigma_1+\frac{1}{2}\sigma_3\sigma_1\right)-\left(\frac{1}{4}\sigma_3^2+\frac{1}{4}\sigma_1^2-\frac{1}{2}\sigma_3\sigma_1\right)\leq 0$$

$$\tau^2+\left[\sigma^2+\left(\frac{1}{4}\sigma_3^2+\frac{1}{4}\sigma_1^2+\frac{1}{2}\sigma_3\sigma_1\right)-\sigma\sigma_3-\sigma\sigma_1\right]-\left(\frac{1}{4}\sigma_3^2+\frac{1}{4}\sigma_1^2-\frac{1}{2}\sigma_3\sigma_1\right)\leq 0$$

$$\tau^2+\left[\sigma-\frac{1}{2}\left(\sigma_3+\sigma_1\right)\right]^2-\left[\frac{1}{2}\left(\sigma_3-\sigma_1\right)\right]^2\leq 0$$

Third inequality:

$$\tau^2+\sigma^2-\sigma\sigma_1-\sigma\sigma_2-\sigma_1\sigma_2\geq 0$$

$$\tau^2+\sigma^2+\frac{1}{4}\sigma_1^2-\frac{1}{4}\sigma_1^2+\frac{1}{4}\sigma_2^2-\frac{1}{4}\sigma_2^2-\sigma\sigma_1-\sigma\sigma_2+\frac{1}{2}\sigma_1\sigma_2+\frac{1}{2}\sigma_1\sigma_2\geq 0$$

$$\tau^2+\left(\sigma^2+\frac{1}{4}\sigma_1^2+\frac{1}{4}\sigma_2^2-\sigma\sigma_1-\sigma\sigma_2+\frac{1}{2}\sigma_1\sigma_2\right)-\left(\frac{1}{4}\sigma_1^2+\frac{1}{4}\sigma_2^2-\frac{1}{2}\sigma_1\sigma_2\right)\geq 0$$

$$\tau^2+\left[\sigma^2+\left(\frac{1}{4}\sigma_1^2+\frac{1}{4}\sigma_2^2+\frac{1}{2}\sigma_1\sigma_2\right)-\sigma\sigma_1-\sigma\sigma_2\right]-\left(\frac{1}{4}\sigma_1^2+\frac{1}{4}\sigma_2^2-\frac{1}{2}\sigma_1\sigma_2\right)\geq 0$$

$$\tau^2+\left[\sigma-\frac{1}{2}(\sigma_1+\sigma_2)\right]^2-\left[\frac{1}{2}(\sigma_1-\sigma_2)\right]^2\geq 0$$

The other two equations can be expanded and compressed similarly. Moreover being σ_2 and σ_3 the two points of the circumference whose segment passes through the center, its half distance gives the radius R, while its mean position gives the x coordinate for the center C. I can use Illustration 12 to apply the same principle to all the three circles.

The general expressions for the radius and the center then are $R_{ij} = \frac{1}{2}(\sigma_i-\sigma_j)$ and $C_{ij} = \frac{1}{2}(\sigma_i+\sigma_j)$

After the substitution the three equations become:

$$\tau^2+(\sigma-C_{23})^2\geq R_{23}^2$$

$$\tau^2+(\sigma-C_{31})^2\leq R_{31}^2$$

$$\tau^2+(\sigma-C_{12})^2\geq R_{12}^2$$

The first of these equations tell us that the point (τ, σ) for the normal and shear stresses on any plane will fall outside the circle with radius R_{23} centered at C_{23} . From the second inequality, indeed, I see that the point (τ, σ) must also lie inside the circle of radius R_{31} centered at C_{31}. Recalling that I have ordered the principal stresses from largest to smallest, this circle is the largest circle that can be formed between any two others. The last inequality says that the point (τ, σ) has to lie outside the circle of radius R_{12} centered at C_{12} . In conclusion, to satisfy all the inequalities, the point (σ, τ) on the Mohr diagram must lie within a region bounded by the three circles between the principal stress values. The transformation of a 3x3 matrix can also be solved by using the *La Place* expansion for matrices. Doing so three 2x2 matrices will be obtained, which can be solved by adopting the standard method for Mohr's circle.

Conclusion

At the end I arrived to the conclusion tensors are extremely useful in mechanics and other branches of physics such as: fluid dynamics, meteorology, molecular dynamics, biology, astrophysics, mechanics, material science and earth science. Tensors themselves can be studied under their mathematical aspect which generally speaking in my essay includes orthonormal transformations and the Mohr's circle which gives a better insight about the application of tensors in mechanics. In my work I analyzed the stress tensor of a body that rotates by a certain angle. Lets suppose an engineer builds a skyscraper in a very windy place, he has to consider the possible rotation of the building and the stress it has to support for not falling apart. This can be done by using the Mohr's circle that shows the stress at any angle and for which angle the maximum stress exists. In some case physicist can encounter data that are inherently symmetric. For non-symmetrical tensors calculations are more complexes and the use graphical representations becomes fundamental. Tensors find applications in modern physics as well. They are one of the main key-stones in general theory of relativity and string theory. I personally believe that in the future they may change our vision of the physical phenomena due to their versatility.

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Kig v. 1.0, Interactive geometry for Ubuntu/Linux